MATHEMATICS

GRADE 10

STUDENT TEXTBOOK

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TOPIC 1

Sets and Operations on Sets

1.1 SET CONCEPT

ACTIVITY 1

As a whole class, arrange yourself in two groups:

(*a*) Group of girls (*b*) Group of boys

Is it possible to make a group of students who weight more than 30 kg? Why?

Discuss the definition of set.

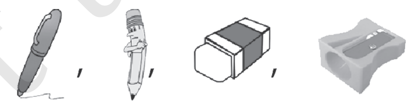
Definition of Set

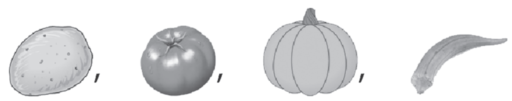
A well-defined collection of objects is called a set. Each object is an element or a member of the set. All the elements of a set are written within the curly brackets { }.

Notation of Set

We use capital letters such as A, B, C, D, ... to name the sets.

For example:

1. A = {  } is a set of geometrical instruments.

2. B = {  } is a set of vegetables.

3. M = {January, February, ..., December} is a set of months of a year.

Let us write a set of days in a week. We write it as:

{Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}

This is well-defined and therefore, is a set.

Now, let us write a set of good football players. We write different sets with different football players, because there is a no particular way to decide who good football players are. Therefore, it is not well-defined and is not a set.

Note: The collection of objects or things should be clearly defined to form a set.

Example 1. *Which of the following are sets?*

(*a*) *{10, 20, 30, 40, 50}*

(*b*) *{prime numbers}*

(*c*) *{all even natural numbers each less than 20 and more than 16}*

(*d*) *The collection of all good wrestlers of the world*

(*e*) *{factors of 24}*

(*f*) *The collection of ten most talented writers of Liberia*

Solution. (*a*), (*b*), (*c*), (*e*)

Note: (*d*) and (*f*) are not well-defined and therefore they are not sets.

Cardinal Number

ACTIVITY 2

Consider the activity 1 again.

Count and write the number of girls.

Count and write the number of boys.

The number of elements in set A is called its cardinal number. We denote it as n(A).

For example:

Look at the set of fruits below:

A = 

Count the number of fruits.

That is, 5.

So, $n(A) = 5$

Example 2. *Determine the cardinality of the set A = {1, 3, 5, 7}*

Solution. It has 4 elements, so, $\boldsymbol{n(A) = 4}$.

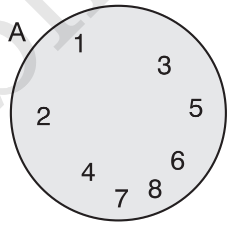
1.2 VENN DIAGRAMS

We can also represent sets using Venn diagrams. In a Venn diagram, the sets are represented by shapes; usually circles or ovals or rectangles. The elements are represented by points in the interior of the shapes.

Example 3. *Given the set A = {1, 2, 3, 4, 5, 6, 7, 8}, draw and label a Venn diagram to represent the set A.*

Solution. Set A = {1, 2, 3, 4, 5, 6, 7, 8}

Draw a circle or an oval. Label it A. Put the elements in A.



Elements of a Set

ACTIVITY 3

In a group of 5 pupils, consider the following sets.

Set of girls (G) = {Islah, Emine, Aaliyah, Shelia, Ella}

Set of boys (B) = {Daniel, Felix, Albert, Amos, William}

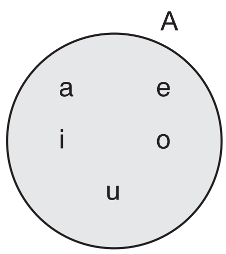
Discuss the following in your group:

1. Does Islah belong to set G?
2. Does William belong to set B?
3. Does Sonia belong to set G?
4. Does Samuel belong to set B?

When an element is a member of a given set, we say that it belongs to that set. We use the symbol $\in$ to show it.

For example:

Suppose set A = {*a*, *e*, *i*, *o*, *u*}



Is *a* an element of set A?

Is *c* an element of set A?

Here, we observe that *a* is an element of set A, but *c* is not an element of set A.

Therefore, $a \in A$

and $c \notin A$.

When an element is not a member of a set, we say that it does not belong to that set. We use the symbol $\notin$ to show it.

Example 4. (*a*) *Write* $\in$ *or* $\notin$ *in the spaces provided.*

(*i*) *3 ___ {multiples of 3}* (*ii*) *x ___ {a, e, i, o, u}*

(*b*) *Use the given Venn diagram and write* $\in$ *or* $\notin$ *in the spaces provided.*

(*i*) *m ___ X* (*ii*) *q ___ X.*

Solution. (*a*) (*i*) 3 $\in$ {multiples of 3} (*ii*) $x \notin \{a, e, i, o, u\}$

(*b*) (*i*) $m \in X$ (*ii*) $q \notin X$

EXERCISE 1.1

1. Which of the following are sets?

(*a*) {10, 9, 8, 7, 6, 5, 4, 3, 2, 1}

(*b*) {2, 4, 6, 8, 10, ..., 100}

(*c*) A collection of all students in the world

(*d*) A collection of the most dangerous animals of the world

(*e*) {factors of 28}

2. Write down all the elements of the following sets.

(*a*) {7, 9, 11, 25, 26}

(*b*) {*a, e, i, o, u*}

(*c*) {the odd numbers between 8 and 25}

(*d*) {the even natural numbers less than 20}

3. Determine the cardinality of each of the following sets.

(*a*) A = {1, 2, 3, 4, 5, 6, 7}

(*b*) B = {2, 4, 6, 8, 10, 12, 14, 16, 20, 22}

(*c*) C = {all factors of 45}

(*d*) P = {all odd natural numbers}

4. If a set has 7 elements, find the cardinality of the set.

5. In the above question 2, represent each set using Venn diagram.

6. Copy and fill in the blanks using the symbol $\in$ or $\notin$.

(*a*) *a* ___ {*a*, *b*, *c*, *d*, *e*}
(*b*) 5 ___ {even prime numbers}
(*c*) 22 ___ {1, 2, 3, ..., 70}
(*d*) 18 ___ {factors of 54}

1.3 TYPES OF SETS

ACTIVITY 4

In the same group as you did activity 3, discuss the following:

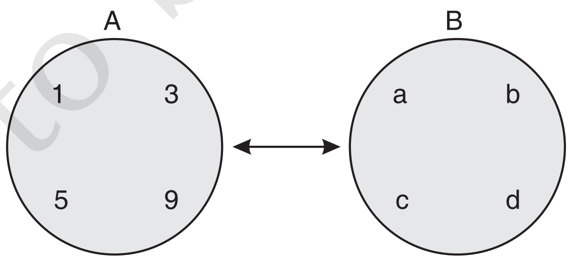
1. Is the number of elements in set G equal to the number of elements in set B? What are these types of sets called?
2. Are the elements in set G exactly the same with the elements in set B? What are these types of sets called?

Equivalent Sets

Sets having equal number of members or elements are called *equivalent sets.*

Consider the sets A = {1, 3, 5, 9} and B = {*a*, *b*, *c*, *d*}. Set A has 4 members. Set B has 4 members. Sets A and B both have the same number of members but do not have exactly the same members. So, they are equivalent sets. We write, Set A $\leftrightarrow$ Set B.

Let us represent them using Venn diagrams.

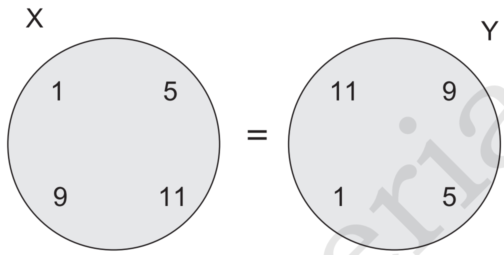


Equal Sets

Sets having exactly the same members or elements are called *equal sets.*

Consider the sets X = {1, 5, 9, 11} and Y = {11, 9, 1, 5}. Set X has 4 members. Set Y has 4 members. Sets X and Y both have exactly the same members. So, they are equal sets. We write, Set X = Set Y.

Let us represent them using Venn diagrams (see the diagram).



Note: Sets having unequal number of members or elements are called *unequal sets.*

Example 5. *State equivalent, equal or unequal sets for each of the following:*

(*a*) *A* = {*1, 2, 5, 6*} *and B* = {*2, 5, 6, 1*}
(*b*) *A* = {*7, 8, 9, 0*} *and B* = {*1, 2, 3, 4*}
(*c*) *X* = {*a, b, c*} *and Y* = {*boat, ship*}

Solution.

(*a*) Set A and set B are equal and equivalent sets.
(*b*) Set A and set B are equivalent and unequal sets.
(*c*) The given sets are unequal sets.

Empty or Null Set

ACTIVITY 5

Is it possible to make a set of persons who have:

(*a*) Three legs (*b*) 3 eyes (*c*) 25 fingers

What do you call these types of sets? Discuss in pairs.

A set that has no member or element is called an *empty* or *null set.*

For example:

X = {The months with 32 days}

Since no month of a year has 32 days,
set X = {The months with 32 days} is an empty set.
The symbol '{ }' or 'φ' is used to denote an empty set.
A set which has at least one element is called a *non-empty set.*

Note: Number 0 does not represent the empty set.
{φ} and {0} are also not empty sets.

Example 6. *Find the empty sets in the following:*

(*a*) *The set of dogs with 6 legs.*

(*b*) *The set of squares with 4 sides.*

(*c*) *The set of cars with 300 doors.*

Solution. Empty sets are: (*a*) and (*c*)

Finite Set

A set with *limited* number of members or one whose last member is known is called a *finite set.*

For example:

(*a*) A = {1, 2, 3, 4, 5, 6, 7, 8}

(*b*) B = {*a*, *b*, *c*, *d*, *e*, *f*, *k*}

(*c*) C = {Men living presently in different parts of the world}

(*d*) D = {Pages in a book}.

We observe that A contains 8 members and B contains 7 members.

How many members does C contain?

It is some natural number which may be quite a big number. D will also contain some fixed number as it's members.

All these sets are *finite.*

Note that the members of a finite set can be counted.

Infinite Set

A set with *unlimited* number of elements or one whose last element is not known is called an *infinite set.*

For example:

(*a*) N = {1, 2, 3, 4, 5, ...} is the set of natural numbers

(*b*) W = {0, 1, 2, 3, 4, ...} is the set of whole numbers

(*c*) E = {2, 4, 6, 8, 10, ...} is the set of even natural numbers.

(*d*) G = {the set of points on a number line}

All these sets are *infinite.*

Note that the members of an infinite set cannot be counted as it continues infinitely.

Example 7. *State which of the following sets are finite or infinite:*

(*a*) {*days of the week*} (*b*) {*odd natural numbers*}

(*c*) {*prime numbers*} (*d*) {*numbers which are multiple of 7*}

(*e*) {*animals living on the earth*}

(*f*) {*months of a year not having 31 days*}.

Solution. (*a*) Finite (*b*) Infinite

(*c*) Infinite (*d*) Infinite

(*e*) Finite (*f*) Finite.

Note: **1.** A set which is empty or consists of a definite number of members is called *finite* otherwise, the set is called *infinite*.

2. A unit set is also called *singleton set*.

3. A unit set is a finite set.

Universal Set

Consider the following set.

U = {Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday}

This is a set of days of a week and is called a universal set.

The set of all members at a time make a universal set.

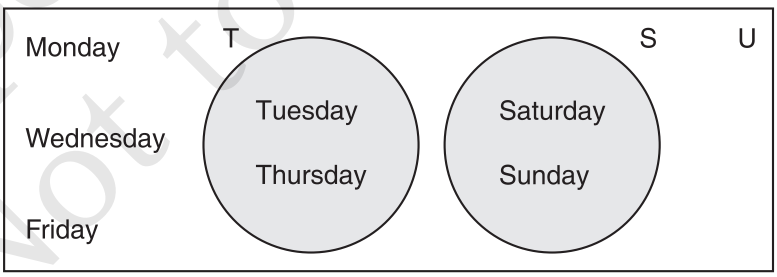
We use the symbol ξ or U to represent it.

We can form several sets using this universal set.

(*a*) A set of days starting with alphabet T = {Tuesday, Thursday}

(*b*) A set of days starting with alphabet S = {Saturday, Sunday}

Let us represent them in a Venn diagram.



Observe that from one universal set, we can form many subsets (*Defined further*).

Example 8. *Three sets are given:*

$A = \{4, 5, 6, 10, p, q, r\}$, $B = \{0, 1, 2, 4, 5, a, b\}$,

$C = \{0, 1, 2, 4, 5, 10, a, b, p, q, r\}$

Is set C the universal set of sets A and B? If not, why?

Solution. No, because 6 is an element of A but not of C.

But C is a universal set of set B because all elements of set B lie in set C.

Example 9. *Consider a universal set of numbers from 1 to 10.*

$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

Write several sets using the above set.

(*a*) *A set of even numbers up to 10*

(*b*) *A set of odd numbers up to 10*

(*c*) *A set of numbers divisible by 3 up to 10.*

Solution.

(*a*) A set of even numbers up to 10 = {2, 4, 6, 8, 10}

(*b*) A set of odd numbers up to 10 = {1, 3, 5, 7, 9}

(*c*) A set of numbers divisible by 3 up to 10 = {3, 6, 9}.

1.4 SUBSETS

ACTIVITY 6

In groups, consider the following sets.

A = {all pupils of your class}

and B = {all girls of your class}

Discuss the following in your group.

1. What is the relationship between set B and set A?
2. Is it true that members of set B are the members of set A?

Consider the sets:

X = {all pupils in your school}

Y = {all pupils in your class}

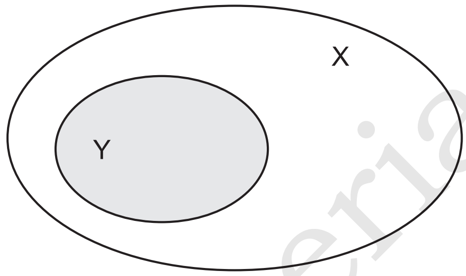
Here every member of set Y is also a member of set X. Therefore, set Y is a subset of set X.

Set Y is a subset of set X is expressed in symbols as $Y \subset X$ or $X \supset Y$. We read it as all elements of set Y are contained in set X.

Using Venn diagram we represent it as (see the adjacent figure):

The symbol '⊂' stands for '*is a subset of*' or '*is contained in*'. If Y is not a subset of X, we write $Y \not\subset X$.

Note: Every set is a subset of itself. Since the empty set ϕ has no element, ϕ is also a subset of every set. Therefore, $A \subset A$ and $\phi \subset A$.



$Y \subset X$

Example 10. *If X = {a, b, c}, then find its subsets.*

Solution. The subsets are:

{ }, {*a*}, {*b*}, {*c*}, {*a*, *b*}, {*b*, *c*}, {*c*, *a*}, {*a*, *b*, *c*}.

Set Notation

The symbols used in set notation are:

Symbols	Meaning	Example
$\in$	belongs to (is an element of)	$b \in \{a, b, c, d, e\}$
$\notin$	does not belong to (is not an element of)	$m \notin \{a, b, c, d, e\}$
$\subset$	is a subset of (is contained in)	$\{a, b, c\} \subset \{a, b, c, d, e\}$
$\not\subset$	is not a subset of (is not contained in)	$\{x, y, z\} \not\subset \{a, b, c, d, e\}$
U	universal set	If A = {*a*, *b*, *c*}, B = {*c*, *d*, *e*}, then U = {*a*, *b*, *c*, *d*, *e*}
{ } or ϕ	null or empty set	A set of months with 32 days is a null set.
=	equal to	If A = {*a*, *b*, *c*}, B = {*c*, *a*, *b*}, then A = B
n(A)	number of elements in the set A	If A = {1, 2, 3, 4, 5}, then *n*(A) = 5

EXERCISE 1.2

1. State as equivalent and equal sets.

(*a*) A = {8, 9, 15} and B = {8, 9, 15}

(*b*) C = {*a*, *d*, *e*, *f*} and D = {1, 2, 3, 4, 5}

(*c*) P = {7, 9, 11, 15} and Q = {15, 11, 9, 7}

(*d*) P = {even numbers less than 10} and
Q = {odd numbers greater than 10}

2. Fill in the following blanks with 'empty set' or 'non-empty set' in your notebook.

(*a*) A cat with six legs ____________

(*b*) A year with fifteen months ____________

(*c*) A car that uses water for fuel ____________

(*d*) Pupils who are owning an airplane ____________

(*e*) A fish that flies ____________

(*f*) A school whose pupils do not study ____________

3. Fill in the following blanks with 'Finite set' or 'Infinite set' in your notebook.

(*a*) P = {5, 10, 15, 20, 25 30} ____________

(*b*) W = {0, 1, 2, 3, ...} *i.e.* set of all counting numbers. ____________

(*c*) N = {1, 2, 3, ...} *i.e.* set of all natural. ____________

(*d*) Q = {natural numbers less than 25} ____________

(*e*) R = {whole numbers between 5 and 45} ____________

4. Given universal set U = {1, 2, 3, 4, ..., 30}.
Write the following sets and represent them using Venn diagrams.

(*a*) A set of numbers less than 10

(*b*) A set of numbers between 10 and 20

(*c*) A set of numbers greater than 6

5. If X = {*a*, *b*}, then find the number of subsets of X.

6. Write the subsets of the set {1, 2, 3}.

7. Consider the sets ϕ, A = {1, 4}, B = {1, 3, 9}, C = {1, 3, 4, 5, 7, 9}.
Insert the symbol $\subset$ or $\not\subset$ between each of the following pairs of sets:

(*a*) ϕ ... B (*b*) A ... B (*c*) A ... C (*d*) B ... C

8. Suppose A = {*a*, *e*, *i*, *o*, *u*} and B = {*a*, *b*, *c*, *d*}. Is A a subset of B? Why? Is B a subset of A? Why?

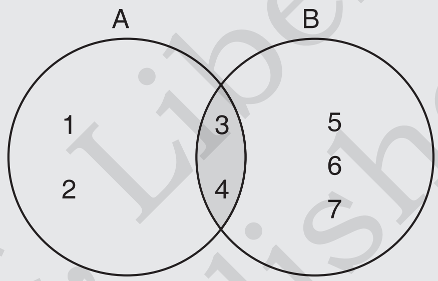
1.5 VENN DIAGRAMS TO ILLUSTRATE INTERSECTION (∩) OF SETS

ACTIVITY 7

In a group of 4 pupils, observe the following Venn diagram.

Discuss the following:

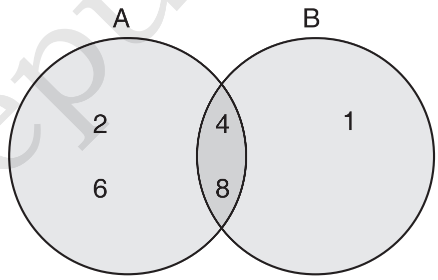
1. What are the elements of set A?
2. What are the elements of set B?
3. What are the elements of set A and set B if combined together?
4. What are the common elements of set A and set B?



The intersection of two sets A and B is the set formed by putting the common elements of these two sets together. The symbol 'Ç' means 'intersection'.

For example:

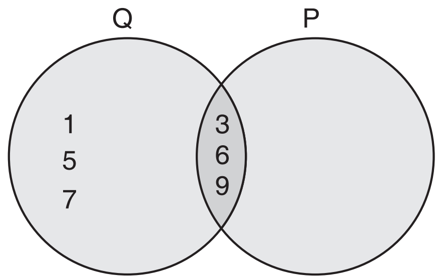
If A = {2, 4, 6, 8} and B = {1, 4, 8}, then $A \cap B$ = {4, 8}, since 4 and 8 belong to both sets A and B. The shaded region in the figure is the intersection of sets A and B.



Example 11. *If P = {3, 6, 9} and Q = {1, 3, 5, 6, 7, 9}, then find* $P \cap Q$.

Solution. We have $P \cap Q$ = {3, 6, 9}

Using Venn diagram it is represented as (see the adjacent figure).



1.6 VENN DIAGRAMS TO ILLUSTRATE UNION (∪) OF SETS

The union of two sets A and B is a set formed by putting the elements of two sets together. The symbol ∪ means 'union'.

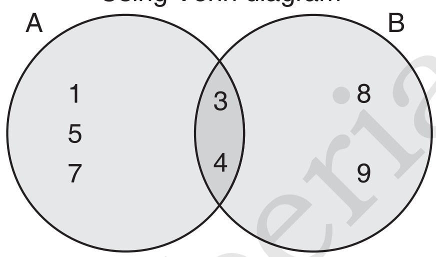
For example:

If A = {1, 3, 4, 5, 7} and B = {3, 4, 8, 9}, then

A ∪ B = {1, 3, 4, 5, 7, 8, 9}

Using Venn diagram it is represented as (see the adjacent figure):

Using Venn diagram



Note: If a number appears in both sets, it is written once.

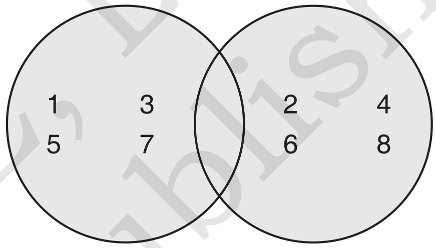
Example 12. *If set A = {1, 3, 5, 7} and B = {2, 4, 6, 8}, find A ∪ B.*

Solution. The union of sets A and B, that is,

A ∪ B = {1, 2, 3, 4, 5, 6, 7, 8}

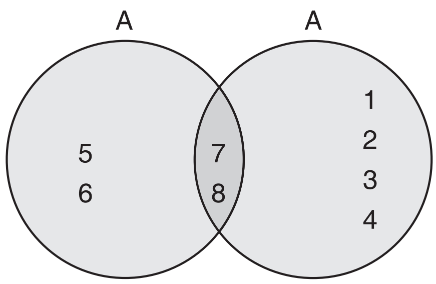
Using Venn diagram it is represented as (see the adjacent figure):

Venn diagram



EXERCISE 1.3

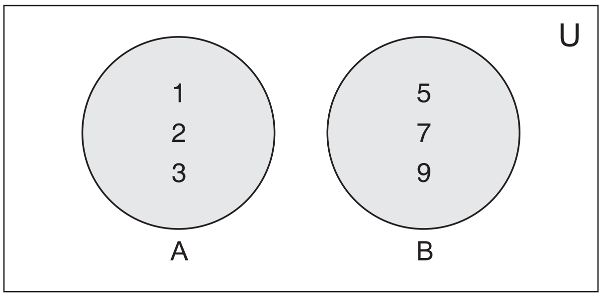
1. If Q = {4, 8, 12, 16} and R = {12, 13, 14}, find Q ∩ R.
2. Find B ∩ C if B = {even numbers less than 10} and C = {multiples of 3 less than 10}
3. If A = {1, 3, 4, 5, 7} and B = {3, 4, 8, 9}, find A ∩ B.
4. If Q = { 7, 8, 9, 10} and R = {5, 6, 7, 8}, find Q ∪ R.
5. If X = {prime numbers less than 13} and Y = {odd natural numbers less than 13}, then
   (*a*) list the members of sets X and Y.
   (*b*) list the members of X ∪ Y.
6. If P = {prime numbers less than 20} and Q = {odd natural numbers less than 10}, find P ∪ Q.
7. Study the given Venn diagram and answer the following questions.
   (*a*) Find the members of sets A and B.
   (*b*) Find A ∪ B.



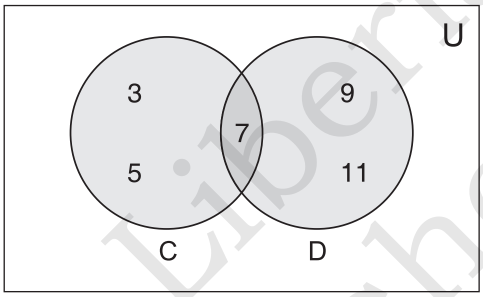
Venn diagram

1.7(A) VENN DIAGRAMS TO SHOW DISJOINT SETS

Two sets A and B are said to be disjoint, if they have no element in common.



Disjoint Sets



Not Disjoint Sets

ACTIVITY 8

Consider the activity 1 again.

The first group is a group of girls.

The second group is a group of boys.

Is there any pupil who is common in both the groups?

Example 13. *If A = {4, 6, 10} and B = {7, 11, 15}, then find A ∩ B. Are the sets A and B disjoint sets? If not, why?*

Solution. Here, $A \cap B = \phi$

As there is no common element in A and B, therefore, these are disjoint sets.

1.7(B) VENN DIAGRAMS TO SHOW COMPLEMENT OF A SET

ACTIVITY 9

Work in pairs. Consider A = {2, 4, 6, 8, 10}; B = {1, 3, 5, 7, 9} and U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}.

1. Find (*i*) U – A (*ii*) U – B
2. What do you call the new sets you get?

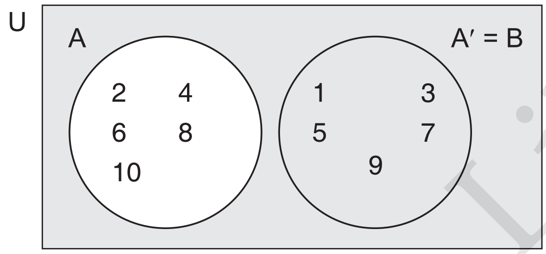
In this activity, set A has all the elements of the universal set U which are not in set B. Similarly, set B has all the elements of the universal set U which are not in set A.

These two sets complement each other with respect to set U.

Symbolically, we write A′ or C_A^U to denote the *complement of A* or A′ = U – A.

Here, A′ = U – A = {1, 3, 5, 7, 9} = B

Let us use Venn diagram to represent A′.



Shaded area in the universal set represents complement of set A.

Example 14. *Find A′ for A = {3, 6, 7, 8}, where the universal set U is given by*

(*a*) *U = {1, 2, 3, ..., 8}* (*b*) *U = {2, 3, 4, ..., 10}*

Solution. We have A = {3, 6, 7, 8}

(*a*) U = {1, 2, 3, ..., 8}

∴ A′ = U – A = Elements of set U which are not in A

= **{1, 2, 4, 5}**

(*b*) U = {2, 3, 4, ..., 10}

∴ A′ = U – A = {2, 3, 4, ..., 10} – {3, 6, 7, 8}

= **{2, 4, 5, 9, 10}**

Example 15. *If the universal set U = {7, 14, 21, 28, 35}, find C_A^U, where*

(*a*) *A = {21, 35}* (*b*) *A = {first two multiples of 7}*

Solution. We have U = {7, 14, 21, 28, 35}

(*a*) A = {21, 35}

∴ C_A^U = U – A = {7, 14, 21, 28, 35} – {21, 35}

= **{7, 14, 28}**

(*b*) A = {first two multiples of 7} = {7, 14}

∴ C_A^U = U – A = {7, 14, 21, 28, 35} – {7, 14}

= **{21, 28, 35}**

Example 16. *If A = {2, 4, 6}; B = {2, 3, 5} and the universal set U = {1, 2, 3, 4, 5, 6, 7, 8}, find (A ∪ B)′ and (A ∩ B)′.*

Solution. $A \cup B = \{2, 4, 6\} \cup \{2, 3, 5\} = \{2, 3, 4, 5, 6\}$

$\therefore \quad (A \cup B)' = U - (A \cup B) = \{1, 2, 3, 4, 5, 6, 7, 8\} - \{2, 3, 4, 5, 6\}$

$= \mathbf{\{1, 7, 8\}}$

Now, $A \cap B = \{2, 4, 6\} \cap \{2, 3, 5\} = \{2\}$

$\therefore \quad (A \cap B)' = U - (A \cap B) = \{1, 2, 3, 4, 5, 6, 7, 8\} - \{2\}$

$= \mathbf{\{1, 3, 4, 5, 6, 7, 8\}}$

EXERCISE 1.4

1. Show that set A = {2, 5, 6} and set B = {4, 7, 8} are disjoint sets.

2. Are the set P = {3, 8, 9} and set Q = {9, 10, 11}, disjoint sets? If not, justify your answer.

3. State whether {*a, e, i, o, u*} and {*a, b, c, d*} are disjoint sets or not.

4. If the universal set is U = {1, 2, 3, ..., 10}, find A′ where

(*a*) A = {1, 2, 3, 4, 5} (*b*) A = {1, 7, 10}

(*c*) A = {1} (*d*) A = {9}

Represent them using Venn diagrams.

5. Given the set of natural numbers as the universal set, write down the complements of the following sets.

(*a*) {even natural numbers}

(*b*) the set of odd natural numbers

(*c*) {factor of 2}

(*d*) the set of natural numbers each more than 19

6. Given universal set U = {1, 2, 3, 4, 5, *a, b, c, d, e*}, find the complements of the following sets. Represent them using Venn diagrams.

(*a*) A = {1, 2, 3, 4, 5} (*b*) B = {*a, b, c, d, e*}

7. Given universal set U = {0, 2, 4, 6, 8, 10, ..., 100}, which of the following pairs are the complements of each other?

(*a*) {2, 4, 6, 8, 10} and {12, 14, 16, 18, ..., 100}

(*b*) {0} and {2, 4, 6, 8, 10, ..., 100}

8. If X = {0, 2, 4, 6}, Y = {2, 4, 8, 16} and universal set U = {0, 2, 4, 6, 8, 10, 12, 14, 16}, then find

(*a*) X′ (*b*) Y′ (*c*) $X \cap Y$

1.8 PROPERTIES OF SETS

1. Commutative Property (*Commutativity*)

Union and intersection of sets satisfy the commutative property.

For any two sets A and B,

$A \cup B = B \cup A$ (Commutative Property of Union)

$A \cap B = B \cap A$ (Commutative Property of Intersection)

For example:

(*a*) Consider A = {1, 2, 3, 4} and B = {2, 4, 6}, then

$A \cup B = \{1, 2, 3, 4\} \cup \{2, 4, 6\} = \{1, 2, 3, 4, 6\}$

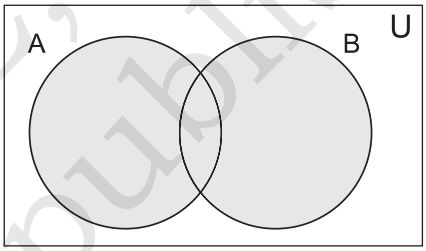
$B \cup A = \{2, 4, 6\} \cup \{1, 2, 3, 4\} = \{2, 4, 6, 1, 3\}$

We observe that $A \cup B$ and $B \cup A$ have exactly the same elements.

Moreover, the elements in a set can be listed in any order.

Therefore, $A \cup B = B \cup A$

The result is true for any two sets. (See the Venn diagram).



A ∪ B (Shaded) = B ∪ A (Shaded)

The Venn diagrams for A ∪ B and B ∪ A are identical (See the Venn diagram).

Thus, union of sets is commutative.

(*b*) Consider A = {*a*, *b*, *c*, *d*, *e*} and B = {*a*, *e*, *i*, *o*, *u*}. A has elements *a* and *e* common with B and B has elements *a* and *e* common with A.

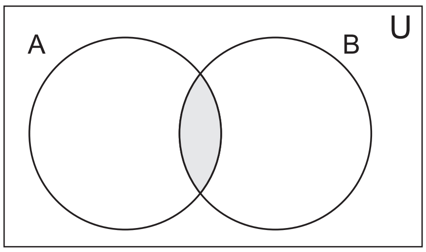
Thus, $A \cap B = \{a, b, c, d, e\} \cap \{a, e, i, o, u\} = \{a, e\}$

$B \cap A = \{a, e, i, o, u\} \cap \{a, b, c, d, e\} = \{a, e\}$

We observe that $A \cap B$ and $B \cap A$ have exactly the same elements.

Therefore, $A \cap B = B \cap A$

The result is true for any two sets (See the Venn diagram).



A ∩ B (Shaded) = B ∩ A (Shaded)

The Venn diagrams for A ∩ B and B ∩ A are identical (See the Venn diagram).

Thus, intersection of sets is commutative.

2. Associative Property (*Associativity*)

Union and intersection of sets satisfy the associative property.

For any three sets A, B and C,

$(A \cup B) \cup C = A \cup (B \cup C)$ (Associative Property of Union)

$(A \cap B) \cap C = A \cap (B \cap C)$ (Associative Property of Intersection)

For example:

(*a*) Consider $A = \{2, 3, 5\}$, $B = \{2, 4, 6\}$ and $C = \{1, 4, 7\}$

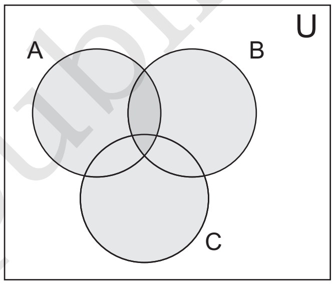
Then $A \cup B = \{2, 3, 5\} \cup \{2, 4, 6\} = \{2, 3, 5, 4, 6\}$

and $(A \cup B) \cup C = \{2, 3, 5, 4, 6\} \cup \{1, 4, 7\} = \{2, 3, 5, 4, 6, 1, 7\}$

Also, $B \cup C = \{2, 4, 6\} \cup \{1, 4, 7\} = \{2, 4, 6, 1, 7\}$

and $A \cup (B \cup C) = \{2, 3, 5\} \cup \{2, 4, 6, 1, 7\} = \{2, 3, 5, 4, 6, 1, 7\}$

We observe that $(A \cup B) \cup C$ and $A \cup (B \cup C)$ have exactly the same elements. Moreover, the elements in a set can be listed in any order.



$(A \cup B) \cup C$ (Shaded) $= A \cup (B \cup C)$ (Shaded)

Therefore, $(A \cup B) \cup C = A \cup (B \cup C)$

The result is true for any three sets.

The Venn diagrams for $(A \cup B) \cup C$ and $A \cup (B \cup C)$ are identical (See the diagram).

Thus, union of sets is associative.

(*b*) Consider $A = \{a, b, c, d\}$, $B = \{b, d, f, g\}$ and $C = \{c, d, e, f\}$

Then, $A \cap B = \{a, b, c, d\} \cap \{b, d, f, g\} = \{b, d\}$

and $(A \cap B) \cap C = \{b, d\} \cap \{c, d, e, f\} = \{d\}$

Also, $B \cap C = \{b, d, f, g\} \cap \{c, d, e, f\} = \{d, f\}$

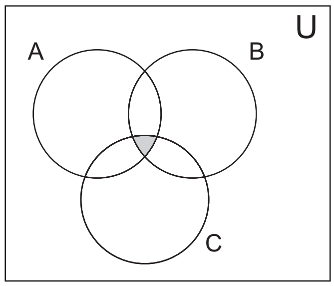
and $A \cap (B \cap C) = \{a, b, c, d\} \cap \{d, f\} = \{d\}$

Clearly, $(A \cap B) \cap C = A \cap (B \cap C)$

The results is true for any three sets.

The Venn diagrams for $(A \cap B) \cap C$ and $A \cap (B \cap C)$ are identical (See the diagram).

Thus, intersection of sets is associative.



$(A \cap B) \cap C$ (Shaded) $= A \cap (B \cap C)$ (Shaded)

3. Distributive Property (*Distributivity*)

Union and intersection of sets satisfy the distributive property.

For any three sets A, B and C,

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

(Union distribute over intersection)

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

(Intersection of distribute over union)

For example:

(*a*) Consider A = {1, 2, 3, 4}; B = {2, 4, 6, 8} and C = {3, 4, 5, 6}.

Then, $B \cap C = \{2, 4, 6, 8\} \cap \{3, 4, 5, 6\} = \{4, 6\}$

and $A \cup (B \cap C) = \{1, 2, 3, 4\} \cup \{4, 6\} = \{1, 2, 3, 4, 6\}$ ...(1)

Also, $A \cup B = \{1, 2, 3, 4\} \cup \{2, 4, 6, 8\} = \{1, 2, 3, 4, 6, 8\}$

and $A \cup C = \{1, 2, 3, 4\} \cup \{3, 4, 5, 6\} = \{1, 2, 3, 4, 5, 6\}$

so that

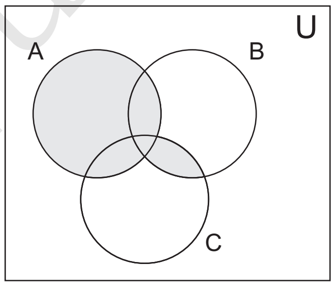
$$(A \cup B) \cap (A \cup C) = \{1, 2, 3, 4, 6, 8\} \cap \{1, 2, 3, 4, 5, 6\} = \{1, 2, 3, 4, 6\} \quad ...(2)$$

From (1) and (2), we conclude that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Thus, the result is true for any three sets.

The Venn diagrams for $A \cup (B \cap C)$ and $(A \cup B) \cap (A \cup C)$ are identical (see the diagram).



$A \cup (B \cap C$ (Shaded)
$= (A \cup B) \cap (A \cup C)$ (Shaded)

Thus, union distributes over intersection.

(*b*) Consider A = {*b*, *d*, *e*}, B = {*a*, *b*, *c*} and C = {*c*, *d*, *f*}.

Then, $B \cup C = \{a, b, c\} \cup \{c, d, f\} = \{a, b, c, d, f\}$

and $A \cap (B \cup C) = \{b, d, e\} \cap \{a, b, c, d, f\} = \{b, d\}$ ...(1)

Also, $A \cap B = \{b, d, e\} \cap \{a, b, c\} = \{b\}$

and $A \cap C = \{b, d, e\} \cap \{c, d, f\} = \{d\}$

so that $(A \cap B) \cup (A \cap C) = \{b\} \cup \{d\} = \{b, d\}$ ...(2)

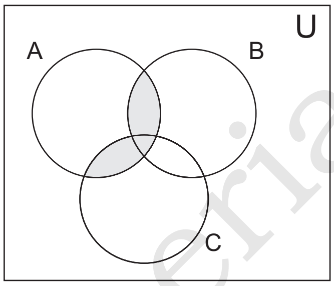
From (1) and (2), we conclude that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Thus, the result is true for any three sets (See the diagram).

The Venn diagrams for $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$ are identical (See the diagram).

Thus, intersection distributes over union.



$A \cap (B \cup C)$ (Shaded)
$= (A \cap B) \cup (A \cap C)$ (Shaded)

4. Some More Properties of Operations on Sets

(*i*) $A \cup \phi = A$ (*ii*) $A \cap \phi = \phi$ (*iii*) $A \cup A = A$

(*iv*) $A \cap A = A$ (*v*) $A \cup U = U$ (*vi*) $A \cap U = A$

(*vii*) $A \subset A \cup B$ (*viii*) $B \subset A \cup B$ (*ix*) $A \cap B \subset A \cup B$

(*x*) $A \cap B \subset A$ (*xi*) $A \cap B \subset B$

(*xii*) If $A \subset B$, then $A \cup B = B$ and $A \cap B = A$

Example 17. *If $A = \{1, 2, 3, 5\}$, $B = \{2, 4, 5\}$ and $C = \{3, 4, 6\}$, find*

(*i*) $A \cup B$ (*ii*) $B \cap C$

(*iii*) $(A \cap B) \cup C$ (*iv*) $A \cap (B \cup C)$

Solution. (*i*) $A \cup B = \{1, 2, 3, 5\} \cup \{2, 4, 5\}$
= Set of all elements of A and B, droping repetitions
$= \{1, 2, 3, 4, 5\}$

(*ii*) $B \cap C = \{2, 4, 5\} \cap \{3, 4, 6\}$
= Set of common elements of A and B = $\{4\}$

(*iii*) $A \cap B = \{1, 2, 3, 5\} \cap \{2, 4, 5\} = \{2, 5\}$
$(A \cap B) \cup C = (2, 5\} \cup \{3, 4, 6\}$
$= \{2, 3, 4, 5, 6\}$

(*iv*) $B \cup C = \{2, 4, 5\} \cup \{3, 4, 6\} = \{2, 3, 4, 5, 6\}$
$A \cap (B \cup C) = \{1, 2, 3, 5\} \cap \{2, 3, 4, 5, 6\} = \{2, 3, 5\}$

Example 18. *If $A = \{a, b, c\}$, $B = \{b, c, d, e\}$ and $C = \{a, d, f\}$. Verify that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.*

Solution. Here $B \cup C = \{b, c, d, e\} \cup \{a, d, f\} = \{a, b, c, d, e, f\}$
$A \cap (B \cup C) = \{a, b, c\} \cap \{a, b, c, d, e, f\} = \{a, b, c\}$ ...(1)

Also, $A \cap B = \{a, b, c\} \cap \{b, c, d, e\} = \{b, c\}$
$A \cap C = \{a, b, c\} \cap \{a, d, f\} = \{a\}$

$(A \cap B) \cup (A \cap C) = \{b, c\} \cup \{a\} = \{a, b, c\}$ ...(2)

From (1) and (2), we have

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

EXERCISE 1.5

1. Let A = {1, 2, 3, 4}, B = {2, 3, 5, 7}, C = {3, 6, 7, 8} and D = {3, 5, 7, 9}; find

(*a*) $A \cup B$ (*b*) $B \cup C$ (*c*) $C \cup D$ (*d*) $A \cap B$

(*e*) $B \cap C$ (*f*) $C \cap D$ (*g*) $A \cup (B \cap D)$ (*h*) $B \cap (A \cup C)$

2. If A = {1, 2, 3, 4, 5, 6}, B = {2, 4, 6, 8} and C = {3, 5, 6, 8}, verify that

(*a*) $A \cup B = B \cup A$ (*b*) $B \cap C = C \cap B$

(*c*) $(A \cup B) \cup C = A \cup (B \cup C)$ (*d*) $(A \cap B) \cap C = A \cap (B \cap C)$

3. Let A = {2, 3, 4} and B = {2, 3, 4, 5, 6}. Use Venn diagram to show:

(*a*) $A \subset B$ (*b*) $A \cap B = A$ (*c*) $A \cup B = B$

4. Let A = {*x* : *x* is a letter of the word PERMANENT}

and B = {*x* : *x* is a letter of the word TEMPORARY}

(*a*) Find $A \cup B$ (*b*) Find $A \cap B$

(*c*) Verify that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

5. Let A = {*x* : *x* is an even natural number}

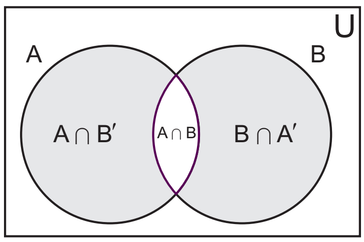
B = {*x* : *x* is an odd natural number}

and C = {*x* : *x* is a prime number}

Find (*a*) $A \cap B$ (*b*) $A \cup B$ (*c*) $A \cap C$.

1.9 VENN DIAGRAMS TO SOLVE TWO-SET *AND* THREE-SET PROBLEMS

If A and B are two finite sets, then it is clear from the given Venn diagram that



(*i*) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(*ii*) $n(A \cap B') = n(A) - n(A \cap B)$

(*iii*) $n(B \cap A') = n(B) - n(A \cap B)$

(*iv*) $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B)$

If A, B and C are three finite sets, then

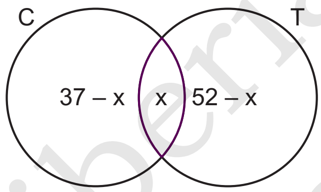
(*i*) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

(*ii*) $n(A \cup B \cup C) = n(A) + n(B) + n(C)$ [if the sets are mutually disjoint]

Example 19. *In a group of 70 persons, 37 like coffee, 52 like tea and each person likes at least one of the two drinks. How many person like (i) both coffee and tea? (ii) coffee and not tea? (iii) tea and not coffee?*

Solution. Let C = Set of persons who like coffee

T = Set of persons who like tea



Suppose x persons like both coffee and tea.

Then,

$n(C \cap T) = x$. Write x in the region common to C and T.

$n(C) = 37 \Rightarrow (37 - x)$ like coffee and not tea. Write $(37 - x)$ in the remaining part of C.

$n(T) = 52 \Rightarrow (52 - x)$ like tea and not coffee. Write $(52 - x)$ in the remaining part of T as shown in the diagram.

Since each person likes at least one of the two drinks, $n(C \cup T) = 70$

$\Rightarrow \quad (37 - x) + x + (52 - x) = 70$

$\Rightarrow \quad 37 + 52 - x + x - x = 70$

$\Rightarrow \quad 89 - x = 70$

$\Rightarrow \quad -x = 70 - 89$

$\Rightarrow \quad -x = -19 \Rightarrow x = 19$

Therefore, 19 persons like both coffee and tea.

Number of persons who like coffee and not tea

$= n(C \cap T') = 37 - x = 37 - 19 = 18$

Number of persons who like tea and not coffee

$= n(T \cap C') = 52 - x = 52 - 19 = 33$

Example 20. *In a class of 60 pupils, 23 play Hockey, 15 play Basketball and 20 play Cricket. 7 play Hockey and Basketball, 5 play Cricket and Basketball, 4 play Hockey and Cricket and 15 pupils do not play any of these games. Find:*

(a) how many play Hockey, Basketball and Cricket?

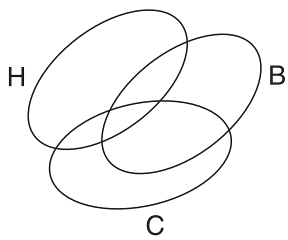
(b) how many play Hockey but not Cricket?

(c) how many play Hockey and Cricket but not Basketball?

Solution. Let H = set of pupil playing Hockey

B = set of pupil playing Basketball

and C = set of pupil playing Cricket



$\therefore \quad n(H) = 23,\ n(B) = 15,\ n(C) = 20,$

$n(H \cap B) = 7,\ n(C \cap B) = 5,$

$n(H \cap C) = 4.$

$n(H' \cap B' \cap C') = 15.$

$\therefore \quad n(H \cup B \cup C) = 60 - n(H' \cap B' \cap C') = 60 - 15 = 45.$

(*a*) We have

$$n(H \cup B \cup C) = n(H) + n(B) + n(C) - n(H \cap B) - n(B \cap C) - n(H \cap C) + n(H \cap B \cap C)$$

$\Rightarrow \quad 45 = 23 + 15 + 20 - 7 - 5 - 4 + n(H \cap B \cap C)$

$\Rightarrow \quad 45 = 42 + n(H \cap B \cap C)$

$\Rightarrow \quad n(H \cap B \cap C) = 45 - 42 = 3$

$\therefore$ No. of pupils playing Hockey, Basketball and Cricket

$= n(H \cap B \cap C) = 3$

(*b*) No. of pupils playing Hockey but not Cricket

$= n(H - C) = n(H) - n(H \cap C) = 23 - 4 =$ **19**

(*c*) No. of pupils playing Hockey and Cricket but not Basketball

$n((H \cap C) - B) = n(H \cap C) - n(H \cap C \cap B)$

$= 4 - 3 =$ **1.**

EXERCISE 1.6

1. Out of 500 car owners investigated, 400 owned cars A and 200 owned cars B; 50 owned both A and B cars. Is this data correct?

2. In a survey of 600 pupils in a school, 150 pupils were found to be drinking Tea and 225 drinking Coffee, 100 were drinking both Tea and Coffee. Find how many pupils were drinking neither Tea nor Coffee.

3. A market research group conducted a survey of 1000 consumers and reported that 720 consumers liked product A and 450 consumers liked product B. What is the least number of consumers that must have liked both products?

4. There are 20 pupils in a Chemistry class and 30 pupils in a Physics class. Find the number of pupils studying either Chemistry or Physics in the following cases:

(*a*) the two classes meet at the same hour

(*b*) the two classes meet at different hours and ten pupils are enrolled in both the courses.

5. Of the number of three athletic teams in a school, 21 are in the basketball team, 26 in hockey team and 29 in the football team. 14 play hockey and basketball, 15 play hockey and football, 12 play football and basketball and 8 play all the games. How many members are there in all?

REVIEW EXERCISE

1. Which of the following are sets?
(*a*) {odd prime numbers}
(*b*) {natural numbers between 40 and 50 not divisible by 2}
(*c*) A collection of all boys in the class
(*d*) {*a*, *b*, *c*, *d*, *e*, *f*}
(*e*) The collection of letters of the word “mathematics”

2. Write down all the elements of the following sets.
(*a*) {the numbers between 10 and 30 not divisible by 3}
(*b*) {the prime numbers between 10 and 50}
(*c*) A set of prime numbers divisible by 3

3. Determine the cardinality of each of the following sets.
(*a*) {all natural numbers}
(*b*) {all prime numbers less than 2}
(*c*) A set of 2-digit numbers divisible by both 3 and 5

4. Copy and fill in the blanks using the symbol $\in$ or $\notin$.
(*a*) 12 ___ {prime numbers} (*b*) 15 ___ {odd prime numbers}
(*c*) 1 ___ {prime numbers} (*d*) 16 ___ {even numbers}

5. State as equivalent and equal sets.
(*a*) M = {*m*, *a*, *t*, *h*, *e*, *i*, *c*, *s*} and N = {*e*, *n*, *g*, *l*, *i*, *s*, *h*}
(*b*) A = {*a*, *e*, *i*, *o*, *u*} and B = {*o*, *u*, *i*, *a*, *e*}
(*c*) A = {10, 20, 30, 40, 50, 60} and B = {70, 80, 90, 100}

6. Fill in the following blanks with ‘empty set’ or ‘non-empty set’ in your notebook.
(*a*) F = {A week having nine days} ____________
(*b*) Q = {Sheep which produce milk} ____________
(*c*) R = {A boy who is as old as his father} ____________
(*d*) X = {A mother of fifty years of age} ____________
(*e*) N = {Fish with wings} ____________
(*f*) Z = {Car which uses diesel as fuel} ____________

7. Fill in the following blanks with 'Finite set' or 'Infinite set' in your notebook.

(*a*) Set of all points in a plane. __________

(*b*) Set of all positive integers which is multiple of 3. __________

(*c*) The set of all persons in Liberia. __________

(*d*) Set of all points in a line segment. __________

(*e*) The set of all birds in California. __________

8. Which of the following is the universal set of the other two?

(*a*) A = {1, 3, 5, 7, 9}, B = {2, 4, 6, 8}, C = {1, 2, 3, 4, 5, 6, 7, 8, 9}

(*b*) A = {P, Q, R, S, T}, B = {P}, C = {Q, R, S, T}

(*c*) A = {pig, goat}, B = {goat, cat, pig, dog}, C = {pig, cat, dog}

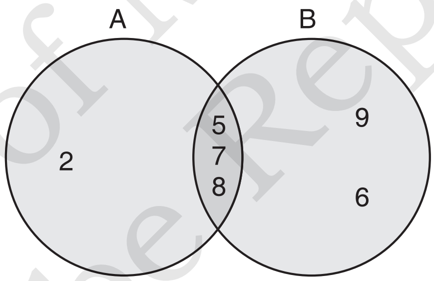
9. Is {all natural numbers} the universal set of {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, ...} and {0}? Why?

10. Is the set C the universal set of sets A and B if

A = {2, 4, 6, 8}, B = {factors of 12}, C = {2, 3, 4, 6, 8, 12}? Why?

11. If P = {7, 9, 13} and Q = {1, 7, 13}, find $P \cap Q$.

12. Study the Venn diagram given below and find $A \cap B$.



13. List the members of each of the sets B = {natural numbers from 20 to 30} and D = {15, 16, 20, 21, 25, 26, 28} and find $B \cup D$.

14. If the universal set is U = {1, 2, 3, …, 10}, find A′ where

(*a*) A = {1, 2, 3, …, 9} (*b*) A = {2, 4, 6, 8, 10}

(*c*) A = {odd numbers up to 9} (*d*) A = {factors of 10}

15. Given universal set U = {1, 2, 3, 4, 5, *a*, *b*, *c*, *d*, *e*}, find the complements of the following sets. Represent them using Venn diagrams.

(*a*) P = {1, 2, 3, *a*, *b*, *c*} (*b*) Q = {4, 5, *d*, *e*}

16. Given universal set U = {0, 2, 4, 6, 8, 10, …, 100}, which of the following pairs are the complements of each other?

(*a*) {multiples of 2 up to 100} and {0}

(*b*) {100, 98, 96, 94, …, 50} and {0, 2, 4, 6, 8, 10, …, 48}

17. If X = {0, 2, 4, 6}, Y = {2, 4, 8, 16} and universal set U = {0, 2, 4, 6, 8, 10, 12, 14, 16}, then find

(*a*) $X \cup Y$ (*b*) $(X \cap Y)'$ (*c*) $(X \cup Y)'$

18. Let A = {1, 2, 3, 4}, B = {2, 3, 5, 7}, C = {3, 6, 7, 8} and D = {3, 5, 7, 9}; find

(*a*) $(A \cup B) \cap (B \cap D)$ (*b*) $(B \cap C) \cup A$.

19. If A = {1, 2, 3, 4, 5, 6}, B = {2, 4, 6, 8} and C = {3, 5, 6, 8}, verify that

(*a*) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (*b*) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

20. There are 200 individuals with a skin disorder. 120 had been exposed to chemical C_1, 50 to chemical C_2, and 30 to both chemicals C_1 and C_2. Find the number of individuals exposed to:

(*a*) chemical C_1 but not chemical C_2

(*b*) chemical C_2 but not chemical C_1

(*c*) chemical C_1 or chemical C_2.

21. Each pupils in a class of 40, studies at least one of the subjects English, Mathematics and Economics. 16 study English, 22 Economics and 26 Mathematics, 5 study English and Economics, 14 Mathematics and Economics and 2 English, Economics and Mathematics. Find the number of pupils who study:

(*a*) English and Mathematics

(*b*) English, Mathematics but not Economics.

MULTIPLE CHOICE QUESTIONS (MCQs)

1. Set A is called ...... of sets B when all the members of set A are also members of set B.

(*a*) the universal set (*b*) the union set

(*c*) the null set (*d*) a subset

2. Two sets which have no common element(s) are known as

(*a*) equal sets (*b*) unequal (*c*) empty sets (*d*) disjoints sets

3. A = {1, 2, 3, 8, 9} and B = {8, 1, x, 3, 2}. If A = B. What is the value of x.

(*a*) 2 (*b*) 3 (*c*) 9 (*d*) 1

4. A = {0, 2, 4, 6} and B = {1, 2, 4, 5}. Find $A \cap B$.

(*a*) {0, 6} (*b*) {2, 4} (*c*) {0, 2, 4} (*d*) {0, 2, 6}

5. Which of the following is true?

(*a*) {2, 3, 5, 7, 27} is a subset of prime numbers.

(*b*) {1, 0, 2, 3, 5} is a subset of odd numbers.

(*c*) {– 2, – 1, 1, 3, 9} is a subset of integers.

(*d*) {0, 2, 6, 9, 12} is a subset of even numbers.

6. P = {Multiples of 3 between 10 and 20}, Q = {even numbers between 10 and 20}. Find P ∩ Q.

(*a*) {12, 18} (*b*) {12, 14, 16, 18}

(*c*) {12, 15, 18} (*d*) {12, 14, 15, 16}

7. R = {1, 3, 5, 7} and S = {2, 4, 6, 8}. Find R ∪ S.

(*a*) {1, 2, 3, 5, 6, 8} (*b*) { }

(*c*) {1, 2, 3, 4, 5, 6, 7, 8} (*d*) {1, 2, 3, 4, 5, 7, 8}

8. If A = {1, 3, 5, 7, 11, 13, 15} and B = {1, 2, 3, 5, 6, 7, 10, 11, 12}. Find A ∩ B.

(*a*) {1, 3, 5, 7, 9, 11} (*b*) {2, 4, 8, 9, 13, 14}

(*c*) {1, 2, 3, 5, 6, 7, 10, 11, 12} (*d*) {1, 3, 5, 7, 11}

9. P = {*g*, *o*, *q*, *s*} and Q = {*h*, *p*, *r*, *t*}. Find P ∪ Q.

(*a*) {*q*, *r*, *s*, *t*} (*b*) {*g*, *h*, *o*, *q*, *r*}

(*c*) {*g*, *h*, *o*, *p*, *q*, *r*, *t*} (*d*) {*g*, *h*, *o*, *p*, *q*, *r*, *s*, *t*}

10. If P = {multiples of 4 less than 16}, find P.

(*a*) {1, 4, 8, 12} (*b*) {4, 8, 2} (*c*) {4, 8, 10} (*d*) {4, 8, 12, 16}

RECAP AT A GLANCE

- A well-defined collection of objects is called a set.
- In a Venn diagram, the sets are represented by shapes; usually circles or ovals or rectangles.
- Sets having exactly the same members or elements are called *equal sets*.
- A set that has no member or element is called an *empty* or *null set*.
- A set with limited number of members or one whose last member is known is called a finite set.
- A set with *unlimited* number of elements or one whose last element is not known is called an *infinite set*.
- Every member of set Y is also a member of set X. Therefore, set Y is a subset of set X.
- The union of two sets A and B is a set formed by putting the elements of two sets together.
- Two sets A and B are said to be disjoint, if they have no element in common.
- Write A′ or C_A^U to denote the *complement of A* or A′ = U – A.
- Union and intersection of sets satisfy the commutative property.
- Union and intersection of sets satisfy the associative property.
- Union and intersection of sets satisfy the distributive property.

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TOPIC 2

Rational Numbers

2.1 ADDITION *AND* SUBTRACTION OF RATIONAL NUMBERS

Identifying Rational Numbers

The word 'rational' comes from the word 'ratio'. Thus, rational number arises as a ratio of two integers (excluding '0' in the denominator).

A rational number is defined as *a number that can be expressed in the form* $\frac{a}{b}$, where a and b are integers and $b \neq 0$.

ACTIVITY 1

Consider $\frac{3}{7}$ is a rational number.

Here, $a = 3$ and $b = 7$.

Is $\frac{-4}{5}$ also a rational number?

For example:

(*a*) $6 = \frac{6}{1}$ (*b*) $0 = \frac{0}{1}$ (*c*) $-8 = -\frac{8}{1}$

(*d*) $\frac{2}{3}$ (*e*) $\frac{-9}{5}$ (*f*) $\frac{7}{13}$

All of these numbers are rational numbers.

Thus, *rational numbers include integers and fractions.*

Example 1. *Identify which of the following are rational numbers and give reasons:*

(*a*) $\frac{2}{7}$ (*b*) $\frac{-4}{3}$ (*c*) $\frac{0}{5}$ (*d*) $\frac{11}{0}$.

Solution. (*a*) $\frac{2}{7}$ is a rational number since 2 and 7 are integers, and $7 \neq 0$.

(*b*) $\frac{-4}{3}$ is a rational number since –4 and 3 are integers and $3 \neq 0$.

(*c*) $\frac{0}{5}$ is a rational number since 0 and 5 are integers and $5 \neq 0$.

(*d*) $\frac{11}{0}$ is not a rational number. Though 11 and 0 are integers, the denominator is zero and division by zero has no meaning.

Addition of Rational Numbers

While adding rational numbers with same denominators, we add the numerators keeping the denominators same.

For example: $\frac{-11}{5} + \frac{7}{5} = \frac{-11+7}{5} = \frac{-4}{5}$

How do we add rational numbers with different denominators?

As in the case of fractions, we first find the LCM of the two denominators. Then, we find the equivalent rational numbers of the given rational numbers with this LCM as the denominator. Then, add the two rational numbers.

For example: Let us add $\frac{-7}{5}$ and $\frac{-2}{3}$.

LCM of 5 and 3 is 15.

So, $\frac{-7}{5} = \frac{-7}{5} \times \frac{3}{3} = \frac{-21}{15}$

and $\frac{-2}{3} = \frac{-2}{3} \times \frac{5}{5} = \frac{-10}{15}$.

Thus, $\frac{-7}{5} + \frac{(-2)}{3} = \frac{-21}{15} + \frac{(-10)}{15} = \frac{-31}{15}$

Subtraction of Rational Numbers

Additive Inverse

What will be $\frac{-4}{7} + \frac{4}{7} = ?$

$$\frac{-4}{7} + \frac{4}{7} = \frac{-4+4}{7} = 0.$$

Also, $\frac{4}{7} + \left(\frac{-4}{7}\right) = 0.$

Similarly, $\frac{-2}{3} + \frac{2}{3} = 0 = \frac{2}{3} + \left(\frac{-2}{3}\right).$

In the case of integers, we call –2 as the additive inverse of 2 and 2 as the additive inverse of –2.

For rational numbers also, we call $\frac{-4}{7}$ as the *additive inverse* of $\frac{4}{7}$ and $\frac{4}{7}$ as the additive inverse of $\frac{-4}{7}$.

ACTIVITY 2

Islah found the difference of two rational numbers $\frac{5}{7}$ and $\frac{3}{8}$ in this way:

$$\frac{5}{7} - \frac{3}{8} = \frac{40}{56} - \frac{21}{56} = \frac{40-21}{56} = \frac{19}{56}$$

Aaliyah knew that for two integers a and b she could write

$$a - b = a + (-b)$$

She tried this for rational numbers also and found out that:

$$\frac{5}{7} - \frac{3}{8} = \frac{5}{7} + \frac{(-3)}{8} = \frac{40}{56} + \frac{(-21)}{56} = \frac{19}{56}.$$

Are Islah and Aaliyah both obtained the same difference?

Try to find $\frac{7}{8} - \frac{5}{9}$ and $\frac{3}{11} - \frac{8}{7}$ in both ways.

Did you get the same answer?

So, we say *while subtracting two rational numbers, we add the additive inverse of the rational number that is being subtracted, to the other rational number.*

Thus, $\frac{5}{3} - \frac{14}{5} = \frac{5}{3}$ + additive inverse of $\frac{14}{5}$

$$= \frac{5}{3} + \frac{(-14)}{5} = \frac{25}{15} + \frac{(-42)}{15} = \frac{-17}{15}.$$

What will be $\frac{2}{7} - \left(\frac{-5}{6}\right)$?

$$\frac{2}{7} - \left(\frac{-5}{6}\right) = \frac{2}{7} + \text{additive inverse of } \left(\frac{-5}{6}\right)$$

$$= \frac{2}{7} + \frac{5}{6} = \frac{12}{42} + \frac{35}{42} = \frac{47}{42}$$

Example 2. *Find:* (*a*) $\frac{-7}{3} + \frac{23}{5}$ (*b*) $\frac{-19}{9} - 6$.

Solution. (*a*) $\frac{-7}{3} + \frac{23}{5}$

LCM of 3 and 5 is 15

So, $\frac{-7}{3} = \frac{-35}{15}$ and $\frac{23}{5} = \frac{69}{15}$

Thus, $\frac{-7}{3} + \frac{23}{5} = \frac{-35}{15} + \frac{69}{15} = \frac{34}{15}$

(*b*) $\frac{-19}{9} - 6 = \frac{-19}{9} - \frac{54}{9} = \frac{-19-54}{9} = \frac{-73}{9}$.

2.2 MULTIPLICATION OF RATIONAL NUMBERS

While multiplying a rational number by an integer (positive or negative), we multiply the numerator by that integer, keeping the denominator unchanged.

Let us now multiply a rational number by a negative integer,

For example: $\frac{-2}{9} \times (-5) = \frac{-2 \times (-5)}{9} = \frac{10}{9}$

Remember, –5 can be written as $\frac{-5}{1}$.

So, $\frac{-2}{9} \times \frac{-5}{1} = \frac{-2 \times (-5)}{9 \times 1} = \frac{10}{9}$

Similarly, $\frac{3}{11} \times (-2) = \frac{3 \times (-2)}{11 \times 1} = \frac{-6}{11}$.

Based on these observations, we find that,

$$\frac{-3}{8} \times \frac{5}{7} = \frac{-3 \times 5}{8 \times 7} = \frac{-15}{56}.$$

So, as we did in the case of fractions, we multiply two rational numbers using the following steps:

Step 1. Multiply the numerators of the two rational numbers.

Step 2. Multiply the denominators of the two rational numbers.

Step 3. Write the product as $\frac{\text{Result of step 1}}{\text{Result of step 2}}$

Thus $\frac{-3}{5} \times \frac{2}{7} = \frac{-3 \times 2}{5 \times 7} = \frac{-6}{35}$.

Also, $\frac{-5}{8} \times \frac{-9}{7} = \frac{-5 \times (-9)}{8 \times 7} = \frac{45}{56}$.

Note **1.** The product of two negative numbers is a positive number, *e.g.,* –2 × –2 = 4.

2. The product of a negative and a positive number is a negative number, *e.g.,* –2 × 2 = –4.

3. The product of two positive numbers is a positive number, *e.g.,* 2 × 2 = 4.

EXERCISE 2.1

1. Is the number $\frac{2}{-3}$ rational? Why?
2. Do rational numbers include all fractions?
3. Is zero a rational number? Can you write it in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$?
4. Find:

   (*a*) $\frac{-13}{7}+\frac{6}{7}$ (*b*) $\frac{19}{5}+\left(\frac{-7}{5}\right)$ (*c*) $\frac{-3}{7}+\frac{2}{3}$ (*d*) $\frac{-5}{6}+\frac{-3}{11}$
5. What will be the additive inverse of the rational numbers given below?

   (*a*) $\frac{-3}{9}$ (*b*) $\frac{-9}{11}$
6. Find: (*a*) $\frac{7}{9}-\frac{2}{5}$ (*b*) $\frac{11}{5}-\frac{(-1)}{3}$.
7. Find: (*a*) $\frac{-3}{5}\times 7$ (*b*) $\frac{-6}{5}\times(-2)$.

2.3 PROPERTIES OF MULTIPLICATION OF RATIONAL NUMBERS

(*a*) **Closure Property:** If a and b are two rational numbers, then $a \times b$ is also a rational number.

For example: For two rational numbers $\frac{3}{4}$ and $\frac{-1}{5}$,

$$\frac{3}{4}\times\frac{(-1)}{5}=\frac{(-3)}{20},$$ which is a rational number.

Hence, rational numbers are *closed* under multiplication.

(*b*) **Associative Property:** If a, b and c are three rational numbers, then

$$a \times (b \times c) = (a \times b) \times c$$

For example: For three rational numbers $\frac{-1}{2}$, $\frac{5}{4}$ and $\frac{-6}{7}$,

$$\frac{-1}{2}\times\left[\frac{5}{4}\times\frac{(-6)}{7}\right]=\frac{-1}{2}\times\frac{(-15)}{14}=\frac{15}{28}$$

and $\left[\frac{-1}{2} \times \frac{5}{4}\right] \times \frac{(-6)}{7} = \frac{-5}{8} \times \frac{(-6)}{7} = \frac{15}{28}$.

Hence, multiplication is *associative* for rational numbers.

(*c*) **Commutative Property:** For two rational numbers a and b,

$$a \times b = b \times a$$

For example: For two rational numbers, $\frac{3}{5}$ and $\frac{(-10)}{11}$

$$\frac{3}{5} \times \frac{(-10)}{11} = \frac{-6}{11} \quad \text{and} \quad \frac{(-10)}{11} \times \frac{3}{5} = \frac{-6}{11}$$

Hence, multiplication is commutative for rational numbers.

(*d*) **Multiplicative Identity:** When a rational number a is multiplied by 1, the product is the rational number itself *i.e.* $a \times 1 = 1 \times a = a$

For example: (*i*) $\frac{-3}{7} \times 1 = \frac{-3}{7}$ (*ii*) $1 \times \frac{4}{5} = \frac{4}{5}$

We say that 1 is the multiplicative *identity* for rational numbers.

(*e*) **Multiplicative Inverse:** The multiplicative inverse for a rational number is its reciprocal.

Let a be a rational number. Then,

$$a \times \frac{1}{a} = 1$$

The product of a rational number and its multiplicative inverse is 1.

Note: 0 has no multiplicative inverse because $\frac{1}{0}$ is not defind.

For example: Consider a rational number $\frac{4}{9}$. Its reciprocal is $\frac{9}{4}$.

$$\therefore \quad \frac{4}{9} \times \frac{9}{4} = 1$$

Remark: Reciprocal of 1 is 1. • Reciprocal of –1 is –1. • Zero has no reciprocal.

(*f*) **Property of 0:** Every rational number when multiplied by 0, gives the product 0.

Hence, for a rational number a

$$a \times 0 = 0$$

Example 3. *Verify:* $\left(\frac{-3}{7} \times \frac{4}{5}\right) \times \frac{-5}{9} = \frac{4}{5} \times \left(\frac{-3}{7} \times \frac{-5}{9}\right)$.

Solution. $\text{LHS} = \left(\frac{-3}{7} \times \frac{4}{5}\right) \times \frac{-5}{9} = \frac{-12}{35} \times \frac{-5}{9} = \frac{4}{21}$ ...(1)

$\text{RHS} = \frac{4}{5} \times \left(\frac{-3}{7} \times \frac{-5}{9}\right) = \frac{4}{5} \times \frac{15}{63} = \frac{4}{21}$ ...(2)

From (1) and (2), LHS = RHS

This proves the associative property of multiplication.

Example 4. *Find the value of x, if* $2 \times (x \times 5) = (2 \times 3) \times 5$.

Solution. By associative property of multiplication, we have

$$a \times (b \times c) = (a \times b) \times c$$

Here, $2 \times (x \times 5) = (2 \times 3) \times 5$

By associative property, we have

$$x = 3$$

Example 5. *Find the multiplicative inverse of the following:*

(*i*) $-1 \times \frac{-2}{5}$ (*ii*) -1

Solution. (*i*) We have $-1 \times \frac{-2}{5} = \frac{2}{5}$

$\therefore$ Multiplicative inverse of $\frac{2}{5}$ is $\frac{5}{2}$

$\because$ $\frac{2}{5} \times \frac{5}{2} = 1$

(*ii*) Multiplicative inverse of –1 is –1.

Example 6. *The product of two rational numbers is* $9\frac{3}{5}$. *If one of them is* $9\frac{3}{7}$, *find the other.*

Solution. Let the 2nd rational number be x.

The 1st rational number = $9\frac{3}{7} = \frac{66}{7}$.

By the given condition,

$$x \times \frac{66}{7} = \frac{48}{5} \qquad \left[\because\ 9\frac{3}{5} = \frac{48}{5}\right]$$

$$\Rightarrow \qquad x = \frac{48}{5} \times \frac{7}{66} = \frac{56}{55} = 1\frac{1}{55}.$$

Hence, the second rational number is $1\frac{1}{55}$.

EXERCISE 2.2

1. Multiplicative inverse of 0 is: ....

2. Multiplicative inverse of $\frac{-3}{5}$ is: ....

3. Write:

(*a*) The rational number that does not have a reciprocal.

(*b*) The rational numbers that are equal to their reciprocals.

4. Find the multiplicative inverse of the following:

(*a*) -13 (*b*) $\frac{-13}{19}$ (*c*) $\frac{1}{5}$ (*d*) $\frac{-5}{8} \times \frac{-3}{7}$

5. Name the property under multiplication used in each of the following:

(*a*) $\frac{-4}{5} \times 1 = 1 \times \frac{-4}{5} = -\frac{4}{5}$ (*b*) $-\frac{13}{17} \times \frac{-2}{7} = \frac{-2}{7} \times \frac{-13}{17}$

6. Verify: $\frac{-5}{9} \times \left(\frac{-2}{5} \times \frac{-3}{7}\right) = \left(\frac{-5}{9} \times \frac{-2}{5}\right) \times \frac{-3}{7}$.

7. Find x in each of the following:

(*a*) $x \times (6 \times 5) = (2 \times 6) \times 5$ (*b*) $5 \times (3 \times 4) = (5 \times 3) \times x$

8. Tell what property allows you to compute $\frac{1}{3} \times \left(6 \times \frac{4}{3}\right)$ as $\left(\frac{1}{3} \times 6\right) \times \frac{4}{3}$.

9. Is $\frac{8}{9}$ the multiplicative inverse of $-1\frac{1}{8}$? Why or why not?

10. Is 0.3 the multiplicative inverse of $3\frac{1}{3}$? Why or why not?

11. By what rational number should $\frac{-8}{39}$ be multiplied to get $\frac{1}{26}$?

2.4 DIVISION OF RATIONAL NUMBERS

We have studied reciprocals of a fraction earlier.

What is the reciprocal of $\frac{2}{7}$*?*

It will be $\frac{7}{2}$. We can extend this idea of reciprocals to rational numbers too.

The reciprocal of $\frac{-2}{7}$ will be $\frac{7}{-2}$ *i.e.,* $\frac{-7}{2}$; that of $\frac{-3}{5}$ would be $\frac{-5}{3}$.

Product of Reciprocals

The product of a rational number with its reciprocal is always 1, *e.g.,* $\frac{a}{b} \times \frac{b}{a} = \frac{ab}{ab} = 1$.

For example: $\frac{-4}{9} \times \left(\text{reciprocal of } \frac{-4}{9}\right) = \frac{-4}{9} \times \frac{-9}{4} = 1$

Similarly, $\frac{-6}{13} \times \frac{-13}{6} = 1$

Try some more examples and confirm this observation.

Let us divide a rational number $\frac{4}{9}$ by another rational number $\frac{-5}{7}$,

That is $\frac{4}{9} \div \frac{-5}{7} = \frac{4}{9} \times \frac{7}{-5} = \frac{-28}{45}$.

We used the idea of reciprocal as done in fractions.

We first divided $\frac{4}{9}$ by $\frac{5}{7}$ and got $\frac{28}{45}$.

That is $\frac{4}{9} \div \frac{-5}{7} = \frac{-28}{45}$.

How did we get that?

We divided them as fractions, ignoring the negative sign and then put the negative sign in the value so obtained.

Both approaches led to the same value $\frac{-28}{45}$. Try dividing $\frac{2}{3}$ by $\frac{-5}{7}$ both ways and see if you will get the same answer.

This shows, *to divide one rational number by the other rational number we multiply the rational number by the reciprocal of the other.*

Thus, $\frac{6}{-5} \div \frac{-2}{3} = \frac{6}{-5} \times \text{reciprocal of} \left(\frac{-2}{3}\right) = \frac{6}{-5} \times \frac{3}{-2} = \frac{18}{10}$

Example 7. *Find:*

(*a*) $\frac{-6}{5} \times \frac{9}{11}$ (*b*) $\frac{-7}{12} \div \left(\frac{-2}{13}\right)$.

Solution. (*a*) $\frac{-6}{5} \times \frac{9}{11} = \frac{-6 \times 9}{5 \times 11} = \frac{-54}{55}$

(*b*) $\frac{-7}{12} \div \left(\frac{-2}{13}\right) = \frac{-7}{12} \times \text{reciprocal of} \left(\frac{-2}{13}\right) = \frac{-7}{12} \times \frac{13}{-2} = \frac{91}{24}$.

EXERCISE 2.3

1. What will be the reciprocal of

(*a*) $\frac{-6}{11}$? (*b*) $\frac{-8}{5}$?

2. Find:

(*a*) $\frac{2}{3} \div \frac{-7}{8}$ (*b*) $\frac{-6}{7} \div \frac{5}{7}$.

2.5 DECIMAL REPRESENTATION

We know that a fraction whose denominator is 10 or 100 or 1000... etc., is called a *decimal fraction.*

Let us express some common fractions as decimal fractions:

Common Fraction	Decimal Fraction
$\frac{2}{10}$	0.2
$\frac{3}{100}$	0.03
$\frac{145}{1000}$	0.145

Identification of Terminating, Non-terminating and Repeating Decimals

A rational number can be expressed as a decimal number in two ways. By long division method or by converting the given rational number into its equivalent rational number whose denominator is 10 or 100 or 1000 ... etc.

For example:

$$\frac{4}{5} = \frac{4\times 20}{5\times 20} = \frac{80}{100} = 0.80$$

$$\frac{5}{8} = \frac{5\times 125}{8\times 125} = \frac{625}{1000} = 0.625$$

The usual method of expressing a rational number in decimals is to carry out the long division process.

Example 8. *Express the following rational numbers in decimal forms:*

(a) $\frac{3}{4}$ *(b)* $\frac{-23}{10}$.

Solution. (*a*) Divide 3 by 4.

$$\begin{array}{r} 4\,)\,3.00\,(\,0.75 \\ -28 \\ \hline 20 \\ -20 \\ \hline 0 \\ \hline \end{array}$$

$\therefore \quad \frac{3}{4} = 0.75$

(*b*) First divide 23 by 10.

$$\begin{array}{r} 10\,)\,23.0\,(\,2.3 \\ -20 \\ \hline 30 \\ -30 \\ \hline 0 \\ \hline \end{array}$$

Thus $\frac{23}{10} = 2.3$

$\therefore \quad \frac{-23}{10} = -2.3$

In the above examples, we get 0 as remainder and there are finite number of digits after the decimal point. Such decimals are called *terminating decimals.*

Example 9. *Express the following rational numbers in decimal form:*

(a) $\frac{10}{3}$ (b) $\frac{1}{7}$.

Solution. (a) Divide 10 by 3

```
3 ) 10.000 ( 3.333
    -9
    ---
     10
     -9
     ---
      10
      -9
      ---
       10
       -9
       ---
        1
```

$\therefore \quad \frac{10}{3} = 3.333...$

(b) Divide 1 by 7.

```
7 ) 1.000000... ( 0.142857
    -7
    ---
     30
    -28
    ---
      20
     -14
     ---
       60
      -56
      ---
        40
       -35
       ---
         50
        -49
        ---
          1
```

$\therefore \quad \frac{1}{7} = 0.142857...$

From the above examples, we observe that:

(a) The division never comes to an end.

(b) The block of digits repeats itself again and again.

We call such decimals a *non-terminating* and *repeating* or *non-terminating* and *recurring decimals.*

While writing in decimals, we place a bar or dot over the repeated part.

For example:

$$\frac{10}{3} = 3.333... = 3.\overline{3} = 3.\dot{3}$$

$$\frac{1}{7} = 0.142857... = 0.\overline{142857} = 0.\dot{1}4285\dot{7}$$

Hence, we can say that *every rational number can be expressed as either a terminating decimal or a non-terminating* and *repeating (recurring) decimal.*

Recognising Decimal Fractions that are Non-terminating and Non-repeating

Let us consider an example of a *non-terminating* and *non-repeating* decimal.

0.10110111011110... is a decimal which does not terminate and has no repeating part. Such numbers are called *non-terminating* and *non-repeating decimals*. They cannot be converted into exact rational numbers. So they are called *non-rational* or *irrational numbers.*

$\sqrt{2}$ and π are also non-rational numbers. Here are their decimal expansions up to a certain stage:

$$\sqrt{2} = 1.4142135623730950488016\,88...$$

$$\pi = 3.14159265358979323846264338...$$

(Note that, we take $\frac{22}{7}$ as an approximate value for π, but $\pi \neq \frac{22}{7}$).

The *square root of a prime number* is an irrational number.

Since, 2, 3, 5, 7, 11, 13, 17, 19, ... are all prime numbers, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt{11}$, $\sqrt{13}$, $\sqrt{17}$, $\sqrt{19}$ are all *irrational numbers.*

Example 10. *Explain why 0.333 is a rational number and π is not.*

Solution. We have, $0.333 = \frac{333}{1000}$

So, it is a rational number.

$$\pi = 3.14159265...$$

It is a decimal which does not terminate and has no repeating part. So, it is not a rational number.

2.6 REAL NUMBERS (R)

Now, equipped with the knowledge of both, the rational as well as the irrational, if we again look at the number line to explore, if any number is left on it. The answer is emphatic no ! It turns out that:

(*i*) The entire collection of all rational numbers and irrational numbers has been picked up, and

(*ii*) No point, on the number line, is now left unrepresented by a number.

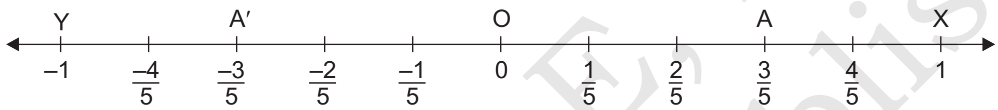
Every real number is either a rational number or an irrational number.

2.7 THE REAL NUMBER LINE

We know that on a number line of integers, the points on the right of zero represent positive integers and the points on the left of zero represent negative integers. Similarly we can represent rational numbers.

Example 11. *Represent* $\frac{3}{5}$ *and* $\frac{-3}{5}$ *on the number line.*

Solution. Draw a number line, represent zero by O. At equal distances from O mark 1 by X and –1 by Y. Divide OX and OY into 5 equal parts.



The point A represents the rational number $\frac{3}{5}$ and the point A′ represents the rational number $\frac{-3}{5}$.

EXERCISE 2.4

1. Express the following rational numbers in decimal form

(*a*) $\frac{47}{40}$ (*b*) $\frac{24}{25}$ (*c*) $\frac{15}{27}$ (*d*) $\frac{12}{13}$

2. Explain why 0.555 is a rational number and π is not.

3. Write any two terminating and two non-terminating rational numbers.

4. Represent each of the following rational numbers on the number line:

(*a*) $\frac{3}{5}$ (*b*) $-1\frac{1}{3}$

5. Represent $\frac{1}{5}, \frac{2}{3}, \frac{-1}{3}, \frac{-1}{5}$ on the same number line.

2.8 PROPERTIES OF REAL NUMBERS

Since, the real number system contains both the systems of rational as well as that of irrational, it is quite natural for it to exhibit their properties also. It, therefore, turns out that:

1. **Algebraic Property:** *Real numbers like rational and irrational— satisfy the commutative, associative and distributive laws for addition (+) and multiplication (×).*
2. **Closure Property:** *Real numbers are closed with respect to, addition, subtraction, multiplication and division (except by zero).* That is, if we add, subtract, multiply or divide (except by zero) two real numbers, we again get a real number.
3. **Denseness Property:** *Between any two different real numbers, there always lies a real number and hence there exist infinitely many real numbers.*
4. **Completeness Property:** *Every real number is represented by a unique point on the number line and conversely, every point on the number line represents a unique real number.*

Operations on Real Numbers

1. *The sum (difference) of a rational number and an irrational number is irrational.*
2. *The product (quotient) of a non-zero rational number with an irrational number is irrational.*

These operations are described as under:

(*a*) The sum of 2 and $\sqrt{3}$ is written as $2 + \sqrt{3}$

(*b*) The product of 2 and $\sqrt{3}$ is written as $2\sqrt{3}$

(*c*) The product of $\sqrt{2}$ with $\sqrt{3}$ is written as $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$

(*d*) The quotient of 7 and $\sqrt{5}$ is written as $\frac{7}{\sqrt{5}}$.

Example 12. *Find a rational number between* $\frac{2}{3}$ *and* $\frac{1}{6}$.

Solution. For this we find the mean of the given rational numbers.

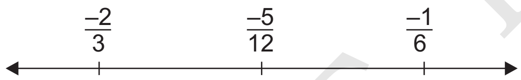
$$\left(\frac{2}{3}+\frac{1}{6}\right) \div 2 = \frac{5}{6} \div 2 = \frac{5}{6} \times \frac{1}{2} = \frac{5}{12}$$

$\frac{5}{12}$ lies between $\frac{2}{3}$ and $\frac{1}{6}$.

Hence, for any two rational numbers a and b, $\frac{a+b}{2}$ is a rational number between them.

Example 13. *Find a rational number between* $\frac{-1}{6}$ *and* $\frac{-2}{3}$.

Solution. A rational number between $\frac{-2}{3}$ and $\frac{-1}{6}$ is



$$\frac{1}{2}\left(\frac{-2}{3}+\frac{-1}{6}\right)=\frac{1}{2}\left(\frac{-4-1}{6}\right)=\frac{1}{2}\times\frac{-5}{6}=\frac{-5}{12}$$

Example 14. *Classify the following numbers as rational or irrational with justification:*

(*i*) $2-\sqrt{5}$ (*ii*) $(3+\sqrt{23})-\sqrt{23}$ (*iii*) $\frac{2\sqrt{7}}{7\sqrt{7}}$

(*iv*) $\frac{1}{\sqrt{2}}$ (*v*) 2π.

Solution. (*i*) $(2-\sqrt{5})$, 2 is rational, $\sqrt{5}$ is irrational.

Since, the difference of a rational and an irrational is irrational.

$\Rightarrow$ $2-\sqrt{5}$ is irrational.

(*ii*) $(3+\sqrt{23})-\sqrt{23}$

Now, $3+\sqrt{23}-\sqrt{23}=3+0=3$, which is rational.

$\Rightarrow$ $(3+\sqrt{23})-\sqrt{23}$ is rational.

(*iii*) $\frac{2\sqrt{7}}{7\sqrt{7}}=\frac{2}{7}$, which is rational.

(*iv*) $\frac{1}{\sqrt{2}}$ is the quotient of a rational and an irrational number.

Since, the quotient of a rational and an irrational number is an irrational number.

$\Rightarrow \quad \frac{1}{\sqrt{2}}$ is an irrational number.

(*v*) 2π = (rational) × (irrational)

2 is a rational number and π is an irrational number.

Since the product of a rational and an irrational is an irrational number.

$\Rightarrow \quad 2\pi$ is an irrational number.

EXERCISE 2.5

1. Insert a rational between each of given pairs of rational numbers.

(*a*) $\frac{1}{4}, \frac{2}{3}$ (*b*) $\frac{-4}{5}, \frac{1}{10}$ (*c*) $\frac{-5}{6}, \frac{-2}{5}$

2. Find three rational numbers between:

(*a*) $\frac{1}{5}$ and $\frac{4}{5}$ (*b*) $\frac{-1}{2}$ and $\frac{3}{4}$

3. Write 9 rational numbers between –1 and 2.

4. Write 10 rational numbers between $\frac{-3}{4}$ and $\frac{5}{6}$.

5. Insert 30 rational numbers between $\frac{2}{5}$ and $\frac{3}{4}$.

6. Classify the following numbers as rational or irrational with justifications:

(*a*) $\sqrt{196}$ (*b*) $3\sqrt{18}$ (*c*) $\sqrt{\frac{9}{27}}$

(*d*) $\left(1+\sqrt{5}\right)-\left(4+\sqrt{5}\right)$ (*e*) 10.124124 .... (*f*) 1.010010001 ....

2.9 APPROXIMATION

Let us consider the numbers 0.684, 9.786 and 0.00849. Now to approximate each of the numbers to two decimal places, we shall have 0.68, 9.79 and 0.01 respectively.

Example 15. *Approximate the number 87354 to the nearest*

(*i*) *10* (*ii*) *100* (*iii*) *1000*

Solution. (*i*) Since the last or ones digit is less than 5, so the second digit from the right side remains the same (*otherwise add 1 to it*) and the ones digit taken as '0'.

∴ The required number round off 10 is 87350.

(*ii*) Since the second digit is equal to 5, so the third digit from the right side will be 1 more *i.e.,* 3 + 1 = 4.

∴ The required number round off 100 is 87400.

(*iii*) Since the third digit is less than 5, so the fourth digit from the right side remains the same.

∴ The required number round off 1000 is 87000.

Rounding Off Numbers Nearest 10, 100, 1000 etc.

- Numbers equal to *or* greater than 5 are rounded up as 10.
- Numbers equal to *or* greater than 50 are rounded up as 100.
- Numbers equal to *or* greater than 500 are rounded up as 1000.
- Numbers equal to *or* greater than half of a given whole number are rounded up to that whole number, otherwise they are taken as zero.

Example 16. *Round off the number 5.3261 to the nearest*

(i) 10 *(ii) 100* *(iii) 1000*

Solution. (*i*) 5.3 (*ii*) 5.33 (*iii*) 5.326

2.10 STANDARD FORM

Consider the mass of the earth. If we write it in numerical form, we have to place 24 zeros to the right of 6 as 60,00,00,00,00,00,00,00,00,00,00,000. This makes reading the number difficult. So we express it as 6×10^{24} kg.

Expressing a number as a product of a numerical value from 1 to less than 10 and the power of 10 is called *standard form.*

For example: The speed of light can be expressed as 3×10^8 m/s.

We can write the powers of 10 as:

$10^0 = 1$; $10^1 = 10$; $10^2 = 100$; $10^3 = 1{,}000$; $10^4 = 10{,}000$;

$10^5 = 100{,}000$ and so on.

In general, the standard form of a number is $x \times 10^n$.

Here, x takes values from 1 to less than 10, and n is a whole number.

We raise the power of 10 to as many times as we shift the decimal point to the left.

For example: Consider a number 380000000000 we write it as 3.8×10^{11}.

We reduce the power of 10 as many times as we shift the decimal point to the right.

For example:

We express a number 0.0052×10^6 as $5.2 \times 10^{6-3}$ or 5.2×10^3.

Example 17. *Express 63,00,000 in standard form.*

Solution. $63,00,000 = 63 \times 1,00,000 = 6.3 \times 10 \times 10^5$

$= 6.3 \times 10^6$

Example 18. *Express 3.627×10^7 in usual form.*

Solution. $3.627 \times 10^7 = \frac{3,627}{1000} \times 10^7 = \frac{3,627 \times 10^7}{10^3} = 3,627 \times 10^{(7-3)}$

$= 3,627 \times 10^4 = 3,627 \times 10,000 = 3,62,70,000$

Example 19. *Express 0.000061 in standard form.*

Solution. $0.000061 = \frac{61}{10,00,000} = \frac{6.1 \times 10}{10^6} = \frac{6.1}{10^5} = 6.1 \times 10^{-5}$

Example 20. *Express 4.23×10^{-6} in usual form.*

Solution. $4.23 \times 10^{-6} = \frac{423}{100} \times 10^{-6} = \frac{423}{10^2 \times 10^6} = \frac{423}{10^8} = 0.00000423$

Example 21. *Charge of an electron is 0.000000000000000000,16 coulomb. Express the charge of an electron in standard form.*

Solution. $0.000000000000000000,16 = 1.6 \times 10^{-19}$

Therefore, the charge of an electron is 1.6×10^{-19} coulomb.

EXERCISE 2.6

1. Approximate the following numbers to the nearest (*i*) 10 (*ii*) 100 (*iii*) 1000
(*a*) 45683 (*b*) 74004 (*c*) 0.0132 (*d*) 0.678154
(*e*) 732.548 (*f*) 0.5832

2. Express the following numbers in standard form:
(*a*) 3,65,000 (*b*) 29,00,000
(*c*) 3,70,000 (*d*) 5,02,00,00,00,00,00,000

3. Express the following numbers in usual form:
(*a*) 5.21×10^4 (*b*) 3.6983×10^7 (*c*) 6.5×10^6

4. Express the following numbers in standard form:
(*a*) 0.00000038 (*b*) 0.0045
(*c*) 0.0008 (*d*) 0.00000000000165

5. Express the following numbers in usual form:
(*a*) 3×10^4 (*b*) 3.467×10^{-3} (*c*) 8.26×10^{-6}

6. Express the number appearing in the following statements in standard form.
(*a*) The distance between the earth and the sun is 1,49,60,00,00,000 m.
(*b*) Thickness of a paper sheet is 0.0005 mm.

2.11 BINARY OPERATIONS

The word '*Bi*' means two. A binary operation on real numbers is a rule which combines two real numbers to produce a *unique real number.* A binary operation is denoted by $*$. The unique real number which a binary operation $*$ associates with two real numbers a and b is denoted by $a * b$.

If $a * b = a + b - ab$, then $*$ associates the unique real number $a + b - ab$ with a and b.

Example 22. *If* $a * b = a + b + 5$, *find* $8 * (-4)$.

Solution. Replacing a by 8 and b by (–4) in the given binary relation, we have

$$8 * (-4) = 8 + (-4) + 5 = 4 + 5 = 9.$$

Example 23. *If $m * n = 2m + n - mn$, find $5 * 3$.*

Solution. Replacing m by 5 and n by 3 in the given binary relation, we have

$$\begin{aligned} 5 * 3 &= 2 \times 5 + 3 - 5 \times 3 \\ &= 10 + 3 - 15 \\ &= 13 - 15 = -2. \end{aligned}$$

EXERCISE 2.7

1. Let $*$ be a binary operation on R. Find
   (*a*) $2 * 4$ if $a * b = 3a + 2b - 1$ (*b*) $3 * 2$ if $a * b = a + 3b^2$
2. Let $*$ be a binary operation on real numbers defined as $a * b = (2a - b)^2$. Find $3 * 5$ and $5 * 3$. Is $3 * 5 = 5 * 3$?
3. If $a * b = 2a + 5b$, find $4 * 3$ and $3 * 4$. Is $4 * 3 = 3 * 4$? Is the operation $*$ commutative?
4. If $m * n = m^2 - mn + n^2$, find $2 * 6$ and $6 * 2$. Is $2 * 6 = 6 * 2$? Is the operation $*$ commutative.

REVIEW EXERCISE

1. List five rational numbers.
2. What will be the additive inverse of the rational number $\frac{5}{7}$?
3. Find:
   (*a*) $\frac{-3}{4} \times \frac{1}{7}$ (*b*) $\frac{2}{3} \times \frac{-5}{9}$.
4. Write:
   The rational number that is equal to its negative.
5. Name the property under multiplication used in $\frac{-19}{29} \times \frac{29}{-19} = 1$.
6. Multiply $\frac{6}{13}$ by the reciprocal of $-\frac{7}{16}$.
7. By what rational number should $\frac{-33}{8}$ be multiplied to get $\frac{-11}{2}$?
8. Express the following rational numbers in decimal forms:
   (*a*) $\frac{42}{9}$ (*b*) $\frac{-1}{37}$.

9. Represent the following rational number on the number line $2\frac{3}{4}$.

10. Classify the following numbers as rational or irrational with justifications:

(*a*) $\frac{\sqrt{28}}{\sqrt{343}}$ (*b*) $-\sqrt{0.4}$ (*c*) $\frac{\sqrt{12}}{\sqrt{75}}$ (*d*) 0.5918

11. Approximate the following numbers to the nearest (*i*) 10 (*ii*) 100 (*iii*) 1000

(*a*) 618712 (*b*) 23871 (*c*) 584.732 (*d*) 19.8972

12. Express the number appearing in the following statements in standard form.

(*a*) The speed of light is 30,00,00,000 m/s.

(*b*) Radius of a red blood cell is 0.000003 mm.

13. Let $*$ be a binary operation on R. Find $6 * 4$ if $a * b = \text{LCM}(a, b)$.

MULTIPLE CHOICE QUESTIONS (MCQs)

1. A rational number can be represented in the form of:

(*a*) $\frac{p}{q}$ (*b*) pq (*c*) $p + q$ (*d*) $p - q$

2. The value of $\frac{1}{2} \times \frac{3}{5}$ is equal to:

(*a*) $\frac{1}{2}$ (*b*) $\frac{3}{10}$ (*c*) $\frac{3}{5}$ (*d*) $\frac{2}{5}$

3. The value of $\frac{1}{2} \div \frac{3}{5}$ is equal to:

(*a*) $\frac{3}{10}$ (*b*) $\frac{3}{5}$ (*c*) $\frac{6}{5}$ (*d*) $\frac{5}{6}$

4. The value of $\frac{1}{2} + \frac{1}{4}$ is equal to:

(*a*) $\frac{3}{4}$ (*b*) $\frac{3}{2}$ (*c*) $\frac{2}{3}$ (*d*) 1

5. The value of $\frac{5}{4} - \frac{8}{3}$ is:

(*a*) $\frac{17}{12}$ (*b*) $-\frac{17}{12}$ (*c*) $\frac{12}{17}$ (*d*) $-\frac{12}{17}$

6. The associative property is applicable to:

(*a*) Addition and subtraction (*b*) Multiplication and division

(*c*) Addition and multiplication (*d*) Subtraction and division

7. The multiplicative identity of a rational number is:

(*a*) 0 (*b*) 1 (*c*) 2 (*d*) –1

8. What is the product of $\frac{2}{9}$ and $\frac{3}{4}$?

(*a*) 1/6 (*b*) 2/3 (*c*) 1/9 (*d*) 1/4

9. What is the reciprocal of 1/9?

(*a*) 9 (*b*) 0 (*c*) 1 (*d*) None of these

10. What is the value of 100 divided by 0?

(*a*) 0 (*b*) 100 (*c*) 1 (*d*) Undefined

11. How many rational numbers are there between $\frac{3}{4}$ and 1?

(*a*) 0 (*b*) 1 (*c*) 2 (*d*) Countless

12. The operation $*$ defined by $a * b = a + b - ab$, calculate $3 * 5$

(*a*) –1 (*b*) –2 (*c*) –4 (*d*) –7

13. Which rational number has no multiplicative inverse?

(*a*) 1 (*b*) 0 (*c*) –1 (*d*) None of these

14. Additive inverse of $\frac{-2}{3}$ is:

(*a*) $\frac{-3}{2}$ (*b*) $\frac{2}{3}$ (*c*) 0 (*d*) $\frac{3}{2}$

15. A rational number between $\frac{1}{2}$ and $\frac{-2}{3}$ is:

(*a*) $\frac{-1}{12}$ (*b*) $\frac{3}{4}$ (*c*) $\frac{7}{6}$ (*d*) 1

16. Multiplicative inverse of $\frac{-5}{-7}$ is:

(*a*) $\frac{-5}{7}$ (*b*) $\frac{5}{7}$ (*c*) $\frac{7}{5}$ (*d*) $\frac{-7}{5}$

RECAP AT A GLANCE

- A rational number is defined as *a number that can be expressed in the form* $\frac{a}{b}$, where a and b are integers and $b \neq 0$.
- The product of two negative numbers is a positive number.
- The product of a negative and a positive number is a negative number.
- The product of two positive numbers is a positive number.

- Rational numbers are *closed* under multiplication.
- Multiplication is *associative* for rational numbers.
- Multiplication is *commutative* for rational numbers.
- 1 is the multiplicative *identity* for rational numbers.
- The product of a rational number and its multiplicative inverse is 1.
- 0 has no multiplicative inverse.
- The product of a rational number with its reciprocal is always 1.
- Every real number is either a rational number or an irrational number.
- Binary operation on real numbers is a rule which combines two real numbers to produce a *unique real number.*

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TOPIC

3

Algebraic Expression

3.1 ALGEBRAIC STATEMENT

In algebra, a statement is basically sentences which are *either* true *or* false. Those mathematical statements which may consist of words and symbols are called algebraic statements.

For example:

(*i*) The square root of a number is 2 is an algebraic mathematical statement.

(*ii*) 3 times the difference of a number and 8

(*iii*) 5 times the sum of a number and 6

(*iv*) The sum of 5 times a number and 6

(*v*) 8 times the difference of twice a number and 6.

3.2 NUMERICAL STATEMENTS

A numerical statement is a *mathematical phrase* or *problem* made up *entirely of numerical characters.* All levels of numerical statements present themselves daily in our lives.

There are statements which have a definite response; *i.e.*, these are *either* 'true' *or* 'false'.

For example:

(*a*) 7 nines is 64   (*b*) $2 + 3 = 5$

(*c*) $4 \times 6 = 10$ (*d*) $20 \div 5 = 4$

(*e*) $\sqrt{49} = 7$ (*f*) $2^4 = 16$

Here the statements (*b*), (*d*), (*e*) and (*f*) are true.

EXERCISE 3.1

1. Write down the algebraic statements from the following algebraic expressions:

(*a*) $3x$ (*b*) $-5y$ (*c*) $-7t + q$ (*d*) $6 - x^2$

(*e*) $100p - 100q$ (*f*) $x - y$

2. Indicate if the following statements are true or false:

(*a*) $2 + 6 = 8$ (*b*) 9 divides 55 (*c*) 4 is a prime number.

(*d*) $7 - 3 = 4$ (*e*) $8^2 = 65$ (*f*) $\sqrt{100} = 10$

3.3 FORMING AN ALGEBRAIC EXPRESSION

Consider an expression: $5x + 3$

1. x is called the literal *or* the variable as its value changes.
2. 3 is called a constant as its value does not change.
3. 5 is called the numerical coefficient of x.
4. $5x + 3$ is called an algebraic expression.

To make this expression, we say that "5 times the number x is added to 3", *i.e.*, $5x + 3$.

Example 1. *Make algebraic expressions of the following statements:*

(*i*) *The sum of 9 and a number x.*

(*ii*) *6 increased by a number y.*

(*iii*) *The product of a number x and 7.*

(*iv*) *Sum of x and y added to 3 times x.*

(*v*) *Product of x and y added to 8.*

Solution.

(*i*) $9 + x$ (*ii*) $6 + y$ (*iii*) $7x$

(*iv*) Sum of x and y gives $x + y$.

Sum of x and y added to 3 times x gives $(x + y) + 3x$.

(*v*) Product of x and y added to 8 gives $xy + 8$.

EXERCISE 3.2

1. Pick the variables and constants in each expression.

(*a*) $7x + 3$ (*b*) $x - y$ (*c*) $6 - x^2$ (*d*) $3x^2 - 2x + y$

(*e*) $8y + xy - 6$

2. Write down the coefficient of each literal:

(*a*) $6x + 2$ (*b*) $-3x - 4$ (*c*) $-4x + 2y$ (*d*) $2x + 3y + 4z$

(*e*) $100p - 100q$

3. Write down the literal coefficient:

(*a*) $3x$ (*b*) $-5y$ (*c*) $-7t$ (*d*) $6xy$

(*e*) $-25x^2 y^2 z^2$ (*f*) $61pq$

4. Form algebraic expressions of the following statements.

(*a*) The sum of 3 and a number *x*.

(*b*) Six less than a number *t*.

(*c*) The product of –5 and a number *x*.

(*d*) A number *y* divided by 16.

(*e*) 3 times the number *x* added to 4 times the number *y*.

3.4(A) EVALUATING ALGEBRAIC EXPRESSION

In order to evaluate an algebraic expression, you must know the exact values for each variable. Then you will simply substitute and evaluate using the order of operations.

Example 2. *Evaluate x + 7 when*

(*i*) $x = 3$ (*ii*) $x = 12$.

Solution. (*i*) To evaluate, substitute 3 for *x* in the expression, and then simplify

	$x + 7$
Substitute	$\mathbf{3} + 7$
Add	10

Thus, 10 is the final answer.

(*ii*)	$x + 7$
Substitute	$\mathbf{12} + 7$
Add	19

Notice that we got different results for part (*i*) and part (*ii*) even though we started with the same expression. This is because the values used for *x* were different. When we evaluate an expression, the value varies depending on the value used for the variable.

Example 3. *Evaluate the expression* $a + \frac{3 + b^3}{2} - c$ *when* $a = 4$, $b = 3$ *and* $c = 8$.

Solution. We have

$a + (3 + b^3)/2 - c$	
$4 + (3 + 3^3)/2 - 8$	Substitute the given values for each values
$4 + (3 + 27)/2 - 8$	We must start the parenthesis and evaluate the power (exponent) ($3^3 = 27$)
$4 + 30/2 - 8$	Evaluate the parenthesis ($3 + 27 = 30$)
$4 + 15 - 8$	Evaluate the division ($30 \div 2 = 15$)
$19 - 8$	Now, addition comes first ($4 + 15 = 19$)
11	Finally, subtraction ($19 - 8 = 11$)

Thus, 11 is the final answer.

3.4(B) RELATIONS BETWEEN TWO ALGEBRAIC EXPRESSIONS

Addition and Subtraction of Algebraic Expressions

To add or subtract algebraic expressions:

(*i*) We write one expression below the other such that the like terms are written one below the other. Then we add or subtract each term. In subtraction, we change the sign of each term of the expression that is to be subtracted. This is called the *Column Method.*

(*ii*) In the horizontal method, for addition, like terms are grouped together and then combined. For subtraction, the sign of each term of the expression to be subtracted is changed and then added.

Example 4. *Add: $13ab - 9cd - xy$ and $12cd - 4ab$.*

Solution.

Column Method:

$$\begin{array}{rrr} 13ab & -\ 9cd & -\ xy \\ -\ 4ab & +\ 12cd & \\ \hline 9ab & +\ 3cd & -\ xy \\ \hline \end{array}$$

Horizontal Method:

$(13ab - 9cd - xy) + (12cd - 4ab)$

$= (13ab - 4ab) + (-9cd + 12cd) - xy$ [Grouping like terms]

$= 9ab + 3cd - xy$

Example 5. *Subtract: $x^3 - 3x - 1$ from $4x^3 - 2x^2 + 5$.*

Solution.

Column Method:

$$\begin{array}{rrrr} 4x^3 & -\ 2x^2 & +\ 0x & +\ 5 \\ x^3 & +\ 0x^2 & -\ 3x & -\ 1 \\ - & - & + & + \\ \hline 3x^3 & -\ 2x^2 & +\ 3x & +\ 6 \\ \hline \end{array}$$

Horizontal Method:

$(4x^3 - 2x^2 + 5) - (x^3 - 3x - 1) = 4x^3 - 2x^2 + 5 - x^3 + 3x + 1$

$= 4x^3 - x^3 - 2x^2 + 3x + 5 + 1$

$= 3x^3 - 2x^2 + 3x + 6$

Multiplication of Algebraic Expressions

- Let us recall:

  The product of two numbers with like signs is positive and the product of two numbers with unlike signs is negative.

  For examples:

  (*i*) $(+4) \times (+5) = +(4 \times 5) = +20$ (*ii*) $(-6) \times (-2) = +(6 \times 2) = +12$

  (*iii*) $(+3) \times (-5) = -(3 \times 5) = -15$ (*iv*) $(-7) \times (+6) = -(7 \times 6) = -42$

- If x is a variable and a and b are positive integers, then

$$x^a \times x^b = x^{(a+b)}$$

For example:

$$x^5 \times x^3 = x^{(5+3)} = x^8$$

1. Multiplication of monomials

Product of monomials = (Product of their numerical coefficients) × (Product of their variable parts)

Examples 6. *Multiply* $2ab^2$ *by* $-3a^2b^3$.

Solution. Product of numerical coefficients = $2 \times (-3) = -6$

Product of their variable parts = $ab^2 \times a^2b^3 = a^3b^5$

The required product = $(-6) \times a^3b^5 = -6a^3b^5$

We can also directly write as

$$2ab^2 \times (-3a^2b^3) = (-3 \times 2)\ (ab^2 \times a^2b^3) = -6a^3b^5$$

2. Multiplication of a polynomial and a monomial

Multiply each term of the polynomial by the monomial using the distributive law.

Let a be a monomial and $(b + c)$ be a binomial. They can be multiplied as follows:

$$a \times (b + c) = (a \times b) + (a \times c)$$

Example 7. *Multiply* $(3x^2y - 8xy + 5y^2)$ *by* $(-2xy^2)$.

Solution. $(-2xy^2) \times (3x^2y - 8xy + 5y^2)$

$$= (-2xy^2) \times (3x^2y) + (-2xy^2) \times (-8xy) + (-2xy^2) \times (5y^2)$$

$$= -6x^3y^3 + 16x^2y^3 - 10xy^4$$

3. Multiplication of two binomials

Let $(a + b)$ and $(c + d)$ be two binomials. They can be multiplied as follows:

$$\begin{aligned}(a + b)(c + d) &= a \times (c + d) + b \times (c + d)\\ &= (a \times c + a \times d) + (b \times c + b \times d)\\ &= ac + ad + bc + bd\end{aligned}$$

Example 8. *Multiply* $(2a + 5b)$ *and* $(3a - 4b)$.

Solution.

$$\begin{aligned}(2a + 5b)(3a - 4b) &= 2a \times (3a - 4b) + 5b \times (3a - 4b)\\ &= (2a \times 3a - 2a \times 4b) + (5b \times 3a - 5b \times 4b)\\ &= 6a^2 - 8ab + 15ab - 20b^2\\ &= 6a^2 + 7ab - 20b^2\end{aligned}$$

Column method of multiplication:

$2a + 5b$			
$3a - 4b$			
$6a^2 + 15ab$			Multiplying by $3a$
	$- 8ab - 20b^2$		Multiplying by $-4b$
$6a^2 + 7ab - 20b^2$			Adding the like terms

Note: The same procedure is applicable for algebraic expressions containing more than two terms.

Example 9. *Multiply* $(x^2 - 5x + 8)$ *and* $(x^2 + 2x - 3)$.

Solution.

$$
\begin{array}{rrrrrl}
x^2 & -\ 5x & +\ 8 & & & \\
\times\ x^2 & +\ 2x & -\ 3 & & & \\
\hline
x^4 & -\ 5x^3 & +\ 8x^2 & & & \text{Multiplying by } x^2 \\
 & +\ 2x^3 & -\ 10x^2 & +\ 16x & & \text{Multiplying by } 2x \\
 & & -\ 3x^2 & +\ 15x & -\ 24 & \text{Multiplying by } -3 \\
\hline
x^4 & -\ 3x^3 & -\ 5x^2 & +\ 31x & -\ 24 & \text{Adding the like terms} \\
\hline
\end{array}
$$

Division of Algebraic Expressions

In division too, the quotient of two numbers with like signs is positive and the quotient of two numbers with unlike signs is negative.

Recall if x is a variable and a and b are positive integers such that $a > b$, then $(x^a \div x^b) = x^{a-b}$

For example: $(x^7 \div x^3) = x^{7-3} = x^4$

1. Division of a monomial by a monomial

Division of a monomial by another monomial

= (Division of their numerical coefficients)

× (Division of their variable parts)

Example 10. *Divide* $6a^3b^2$ *by* $-2ab$.

Solution. $\dfrac{6a^3b^2}{-2ab} = \left(\dfrac{6}{-2}\right) a^{(3-1)}\ b^{(2-1)} = -3a^2b$

Example 11. *Divide* $-12a^4b^3c$ *by* $-4a^2b^2c$.

Solution. $\dfrac{-12a^4b^3c}{-4a^2b^2c} = \left(\dfrac{-12}{-4}\right)a^{(4-2)}b^{(3-2)}c^{(1-1)}$

$= 3a^2bc^0 = 3a^2b$ $\qquad [\because \; c^0 = 1]$

2. Division of a polynomial by a monomial

Each term of the polynomial is divided by the monomial.

Example 12. *Divide* $(15x^4y^2 + 12x^2y^2 - 9x^3y)$ *by* $(-3xy)$

Solution. $(15x^4y^2 + 12x^2y^2 - 9x^3y) \div (-3xy)$

$= \dfrac{15x^4y^2}{-3xy} + \dfrac{12x^2y^2}{-3xy} - \dfrac{9x^3y}{-3xy}$

$= -5x^{4-1}y^{2-1} - 4x^{2-1}y^{2-1} + 3x^{3-1}y^{1-1}$

$= -5x^3y - 4xy + 3x^2$ $\qquad [\because \; y^0 = 1]$

3. Division of a polynomial by a polynomial

Example 13. *Divide* $12x^2 + 7xy - 12y^2$ *by* $3x + 4y$.

Solution.

Step 1: Arrange the terms of divisor and the dividend in descending order of their variable parts.

$$
\begin{array}{r|l}
 & 4x - 3y \\
\hline
3x + 4y & 12x^2 + 7xy - 12y^2 \\
 & \underset{-}{+}12x^2 \underset{-}{+} 16xy \qquad \text{[Changing signs]} \\
\hline
 & \qquad -9xy - 12y^2 \\
 & \qquad -9xy - 12y^2 \\
 & \qquad \;\; + \quad\; 0 + \quad 0 \qquad \text{[Changing signs]} \\
\hline
 & \qquad\qquad 0 \\
\end{array}
$$

Step 2: Divide the first term of the dividend by the first term of the divisor to get the first term of the quotient, *i.e.*, $\dfrac{12x^2}{3x} = 4x$.

Step 3: Multiply all the terms of the divisor by the first term of the quotient and subtract the result from the dividend.

Step 4: Consider the remainder as the new dividend.

Step 5: Repeat the steps 2, 3 and 4 till the remainder obtained is zero or a polynomial of a degree less than that of the divisor.

Hence, $(12x^2 + 7xy - 12y^2) \div (3x + 4y) = 4x - 3y$.

EXERCISE 3.3

1. Find the value of algebraic expression by substituting the value of literals as given:

(*a*) $9xy + 4$ if = 2, $y = \frac{1}{3}$

(*b*) $a + b + c + a^2 + b^2$, if $a = 2$, $b = -3$, $c = 1$

(*c*) $x^2 - xy + y^2$, if $x = -2$, $y = 3$

2. Add the following algebraic expressions:

(*a*) $3x^2 - 5xy - 1 + 8y^2$; $-2xy + 3y^2 - 5 + 10x^2$; $8 - xy + x^2 - y^2$

(*b*) $2ax - 6by + 4cz$; $4by - 14ax$; $9cz - 4ax - 6by$

3. Subtract the following expressions:

(*a*) $4p^2 + 5q^2 + 7$ from $-4q^2 - 5r^2 - 6$

(*b*) $3x^2 - 6x + 4$ from $5 + x - 2x^2$

4. The perimeter of a triangle is $9y^2 - 9y + 4$ and its two sides are $2y^2 - 3y$ and $4y^2 + 6$. Find its third side.

5. Multiply:

(*a*) $(6a^2) \times (-4ab)$ (*b*) $\left(\frac{1}{3}xy\right) \times \left(-\frac{6}{7}x^2y\right)$

6. Multiply:

(*a*) $(-3x) \times (6x + 5)$ (*b*) $(5ab) \times (3a^2 - 4ab - b^2)$

7. Multiply:

(*a*) $(5a - 4) \times (7a + 5)$ (*b*) $(x^2 + x + 1) \times (1 - x)$

8. Multiply:

(*a*) $(2p^2 - 2pq + 3q^2) \times (3p^2 + 4pq + q^2)$

(*b*) $(x^3 - 5x^2 + 3x + 1) \times (x^2 - 3)$

9. Simplify:

(*a*) $3a(a + b) - 4b(a + b) + 2(ab + b^2)$

(*b*) $(x + y)(x + y + z) - (x - y)(x - y - z)$

10. Divide:

(*a*) $-18a^3b^2$ by $3ab$ (*b*) $75x^4y^2$ by $-25x^3y^2$

(*c*) $-20p^6q^5$ by $-4p^3q^2$

11. Divide:

(*a*) $12a^2b^4 + 16a^3b^3 - 20a^4b^2$ by $4a^2b^2$

(*b*) $8x^3yz - 24xy^3z + 48xyz^3$ by $(-8xyz)$

(*c*) $3abc - 6a^2bc + 4a^2b^2c^2$ by $(-3abc)$

12. Divide:

(*a*) $(x^4 + x^2y^2 + y^4)$ by $(x^2 - xy + y^2)$

(*b*) $(-8a^4 + 16a^3 - a + 2)$ by $(-2a^2 + 3a + 2)$

(*c*) $(16 + 8x + x^6 - 2x^4 + x^2)$ by $(x + 4 - x^3)$

3.5 EXPANSION

Consider the expression $2(x + 3)$. We say that 2 is the *coefficient* of the expression in the brackets. We can *expand* the brackets using the *distributive law*:

$$a(b + c) = ab + ac$$

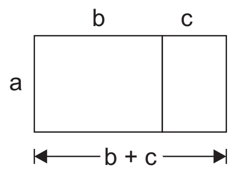
The distributive law says that we must multiply the coefficient by each term within the brackets, and add the results.

Geometric Demonstration:

The overall area is $a(b + c)$.

However, this could also be found by adding the areas of the two small rectangles: $ab + ac$.

So, $a(b + c) = ab + ac$. {equating areas}



Example 14. *Expand the following:*

(*a*) $3(4x + 1)$ (*b*) $2x(5 - 2x)$ (*c*) $x(2x - 1) - 2x(5 - x)$

Solution. (*a*) $3(4x + 1) = 3 \times 4x + 3 \times 1$

$= 12x + 3$

(*b*) $2x(5 - 2x) = 2x\,(5 - 2x)$

$= 2x \times 5 - 2x \times 2x = 10x - 4x^2$

(*c*) $x(2x - 1) - 2x(5 - x) = 2x^2 - x - 10x + 2x^2$

$= 4x^2 - 11x$

EXERCISE 3.4

1. Expand and simplify:

(*a*) $3(x + 1)$ (*b*) $2(5 - x)$ (*c*) $-(x + 2)$

(*d*) $-(3 - x)$ (*e*) $4(a + 2b)$ (*f*) $3(2x + y)$

2. Expand and simplify:

(*a*) $1+2(x+2)$ (*b*) $13-4(x+3)$ (*c*) $3(x-2)+5$

(*d*) $4(3-x)-10$ (*e*) $x(x-1)+x$ (*f*) $2x(3-x)+x^2$

3. Expand and simplify:

(*a*) $3(x-4)+2(5+x)$ (*b*) $2a+(a-2b)$

(*c*) $2a-(a-2b)$ (*d*) $3(y+1)+6(2-y)$

3.6 ALGEBRAIC FRACTIONS

Algebraic fractions are fractions that contain at least one variable either in numerator or in denominator or in both such as:

x is the numerator $\frac{x}{18}$	An expression in terms of x is the denominator $\frac{4}{x+2}$	The numerator is a multiple of x $\frac{2x}{15}$
Both the numerator and the denominator contains an x term $\frac{x+1}{2x}$	Both the numerator and the denominator contain an expression with x $\frac{3x+4}{2x-5}$	The numerator and the denominator are quadratic expressions $\frac{(3x+5)^2}{x^2-4}$

Adding and Subtracting Algebraic Fractions

When adding and subtracting fractions, we must ensure that we have the same denominator.

Example 15. *Calculate:* $\frac{2}{3}-\frac{y}{18}$.

Solution.

$$\frac{2}{3}-\frac{y}{18}=\frac{2\times 18}{54}-\frac{3y}{54}$$

$$=\frac{36}{54}-\frac{3y}{54}=\frac{36-3y}{54}=\frac{3(12-y)}{54}=\frac{12-y}{18}.$$

Example 16. *Calculate:* $\frac{x}{y}+\frac{y}{x}$.

Solution.

$$\frac{x}{y}+\frac{y}{x}=\frac{x^2}{xy}+\frac{y^2}{xy}=\frac{x^2+y^2}{xy}.$$

Example 17. *Simplify:* $\frac{2}{x} - \frac{5}{x+2}$.

Solution. $$\frac{2}{x} - \frac{5}{x+2} = \frac{2(x+2)}{x(x+2)} - \frac{5x}{x(x+2)} = \frac{2x+4-5x}{x(x+2)} = \frac{4-3x}{x(x+2)}.$$

EXERCISE 3.5

Simplify the following fractions. Write your answers in the lowest term.

1. $\frac{x}{3} + \frac{y}{3}$

2. $\frac{9}{8x} + \frac{6}{8x}$

3. $\frac{7}{8x} + \frac{5}{8x} - \frac{3}{8x}$

4. $\frac{12x-15}{12x} - \frac{9x-6}{12x}$

5. $\frac{ab}{4} + \frac{ab}{5}$

6. $\frac{7x}{8} + \frac{3x}{10} - \frac{x}{5}$

7. $\frac{2}{x^2} - \frac{3}{x^2} + \frac{7}{x}$

8. $\frac{x+y}{8} - \frac{x+y}{10}$

3.7 FACTORISATION

Factorisation is the process of writing an expression as a *product* of its *factors*.

Factorisation is the reverse process of expansion.

In *expansions* we have to *remove brackets*, whereas in *factorisation* we have to *insert brackets*. Notice that $3(x + 2)$ is the *product of two factors*, 3 and $x + 2$.

The brackets are essential, since, in $3(x + 2)$ the whole of $x + 2$ is multiplied by 3 whereas in $3x + 2$ only the x is multiplied by 3.

Expansion

$3(x + 2) = x + 6$

Factorisation

To factorise an algebraic expression involving a number of terms we look for the Highest Common Factor (HCF) of the terms and write it in front of a set of brackets. We then find the contents of the brackets.

For example: $5x^2$ and $10xy$ have HCF $5x$.

So, $$5x^2 + 10xy = 5x \times x + 5x \times 2y = 5x(x + 2y).$$

Factorise Fully

Notice that $4a + 12 = 2(2a + 6)$ is not fully factorised as $(2a + 6)$ still has a common factor of 2 which could be removed. Although 2 is a common factor it is not the *highest* common factor. The HCF is 4 and so

$$4a + 12 = 4(a + 3) \text{ is fully factorised.}$$

Example 18. *Fully factorise:*

(*a*) $3a + 6$ (*b*) $ab - 2bc$

Solution. (*a*) $3a + 6 = 3 \times a + 3 \times 2 = 3(a + 2)$ [HCF is 3]

(*b*) $ab - 2bc = a \times b - 2 \times b \times c = b(a - 2c)$ [HCF is b]

Example 19. *Fully factorise:*

(*a*) $8x^2 + 12x$ (*b*) $3y^2 - 6xy$

Solution. (*a*) $8x^2 + 12x = 2 \times 4 \times x \times x + 3 \times 4 \times x$

$= 4x(2x + 3)$ [HCF is $4x$]

(*b*) $3y^2 - 6xy = 3 \times y \times y - 2 \times 3 \times x \times y$

$= 3y(y - 2x)$ [HCF is $3y$]

Exmaple 20. *Fully factorise:*

(*a*) $2(x + 3) + x(x + 3)$ (*b*) $x(x + 4) - (x + 4)$

Solution. (*a*) $2(x + 3) + x(x + 3)$ [HCF is $(x + 3)$]

$= (x + 3)(2 + x)$

(*b*) $x(x + 4) - (x + 4) = x(x + 4) - 1(x + 4)$

$= (x + 4)\ (x - 1).$ [HCF is $(x + 4)$]

Example 21. *Fully factorise $(x - 1)\ (x + 2) + 3(x - 1)$.*

Solution. $(x - 1)[(x + 2) + 3]$ [HCF of $(x - 1)$]

$= (x - 1)\ [(x + 2) + 3]$

$= (x - 1)(x + 5).$

EXERCISE 3.6

1. Copy and complete:

(*a*) $2x + 4 = 2(x + \ldots)$ (*b*) $3a - 12 = 3(a - \ldots)$

(*c*) $15 - 5p = 5(\ldots - p)$ (*d*) $18x + 12 = 6(\ldots + 2)$

2. Copy and complete:

(*a*) $4x + 16 = 4(\ldots + \ldots)$ (*b*) $10 + 5d = 5(\ldots + \ldots)$

(*c*) $5c - 5 = 5(\ldots - \ldots)$ (*d*) $cd + de = d(\ldots + \ldots)$

3. Fully factorise:

(*a*) $3a + 3b$ (*b*) $8x – 16$ (*c*) $3p + 18$ (*d*) $28 – 14x$

4. Fully factorise:

(*a*) $x^2 + 2x$ (*b*) $5x – 2x^2$ (*c*) $4x^2 + 8x$ (*d*) $14x – 7x^2$

5. Fully factorise:

(*a*) $– 9a + 9b$ (*b*) $– 3 + 6b$ (*c*) $– 8a + 4b$ (*d*) $–7c + cd$

6. Fully factorise:

(*a*) $–6a – 6b$ (*b*) $–4 – 8x$ (*c*) $–3y – 6z$ (*d*) $– 9c – cd$

7. Fully factorise:

(*a*) $2(x – 7) + x(x – 7)$ (*b*) $a(x + 3) + b(x + 3)$

(*c*) $4(x + 2) – x(x + 2)$ (*d*) $x(x + 9) + (x + 9)$

8. Fully factorise:

(*a*) $(x + 3)(x – 5) + 4(x + 3)$ (*b*) $5(x – 7) + (x – 7)(x + 2)$

(*c*) $(x + 6)(x + 4) – 8(x + 6)$ (*d*) $(x – 2)^2 – 6(x – 2)$

3.8 PRODUCT OF TWO BINOMIALS

ACTIVITY 1

Consider the figure alongside:

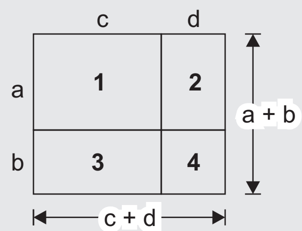
Give an expression for the area of:

(*a*) rectangle 1 (*b*) rectangle 2

(*c*) rectangle 3 (*d*) rectangle 4

(*e*) the overall rectangle.

What can you conclude?



The product $(a + b)(c + d)$

Consider the product $(a + b)(c + d)$.

It has two *factors,* $(a + b)$ and $(c + d)$.

We can evaluate this product by using the distributive law several times.

$$(a + b)(c + d) = a(c + d) + b(c + d)$$
$$= ac + ad + bc + bd$$

So, $$(a + b)(c + d) = ac + ad + bc + bd$$

Example 22. *Expand and simplify: (x + 3) (x + 2).*

Solution. $(x + 3)(x + 2) = x \times x + x \times 2 + 3 \times x + 3 \times 2$

$= x^2 + 2x + 3x + 6 = x^2 + 5x + 6.$

Example 23. *Expand and simplify:*

(*a*) $(x + 3)(x - 3)$ (*b*) $(3x - 5)(3x + 5)$.

Solution. (*a*) $(x + 3)(x - 3) = x^2 - 3x + 3x - 9 = x^2 - 9.$

(*b*) $(3x - 5)(3x + 5) = 9x^2 + 15x - 15x - 25 = 9x^2 - 25$

Example 24. *Expand and simplify:*

(*a*) $(3x + 1)^2$ (*b*) $(2x - 3)^2$

Solution. (*a*) $(3x + 1)^2 = (3x + 1)(3x + 1)$

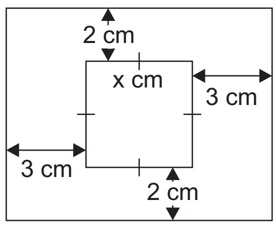
$= 9x^2 + 3x + 3x + 1 = 9x^2 + 6x + 1$

(*b*) $(2x - 3)^2 = (2x - 3)(2x - 3)$

$= 4x^2 - 6x - 6x + 9 = 4x^2 - 12x + 9.$

EXERCISE 3.7

1. Expand and simplify:
   (*a*) $(x + 3)(x + 7)$ (*b*) $(x + 5)(x - 4)$
   (*c*) $(x - 3)(x + 6)$ (*d*) $(x + 2)(x - 2)$
2. Expand and simplify:
   (*a*) $(x + 2)(x - 2)$ (*b*) $(a - 5)(a + 5)$
   (*c*) $(4 + x)(4 - x)$ (*d*) $(2x + 1)(2x - 1)$
3. Expand and simplify:
   (*a*) $(x + 3)^2$ (*b*) $(x - 2)^2$ (*c*) $(3x - 2)^2$ (*d*) $(1 - 3x)^2$
   (*e*) $(3 - 4x)^2$ (*f*) $(5x - y)^2$
4. A square photograph has sides for length x cm. It is surrounded by a wooden frame with the dimensions shown. Show that the area of the rectangle formed by the outside of the frame is given by A = $(x^2 + 10x + 24)$ cm^2.



3.9 DIFFERENCE OF TWO SQUARES

a^2 and b^2 are perfect squares and so $a^2 - b^2$ is called the *difference of two squares.* Notice that $(a + b)(a - b) = a^2 \underbrace{- ab + ab}_{\text{The middle two terms add to zero}} - b^2 = a^2 - b^2$

Thus, $(a + b)(a - b) = a^2 - b^2$

Geometric Demonstration:

Consider the figure alongside:

The shaded area

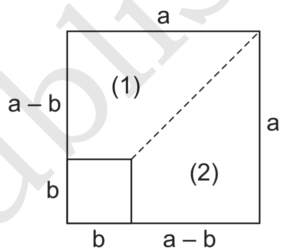
= area of large square – area of small square

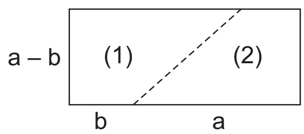
= $a^2 - b^2$

Cutting along the dotted line and flipping (2) over, we can form a rectangle.

The rectangle's area is $(a + b)(a - b)$.

$\therefore \quad (a + b)(a - b) = a^2 - b^2$.





Example 25. *Expand and simplify:*

(*a*) $(x + 5)(x - 5)$ (*b*) $(3 - y)(3 + y)$

(*c*) $(3x + 4y)(3x - 4y)$

Solution. (*a*) $(x + 5)(x - 5) = x^2 - 5^2 = x^2 - 25$

(*b*) $(3 - y)(3 + y) = 3^2 - y^2 = 9 - y^2$.

(*c*) $(3x + 4y)(3x - 4y) = (3x)^2 - (4y)^2 = 9x^2 - 16y^2$.

3.10 PERFECT SQUARE EXPANSION

ACTIVITY 2

Consider the figure alongside:

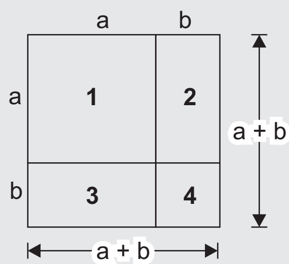
Give an expression for the area of:

(*a*) square 1 (*b*) rectangle 2

(*c*) rectangle 3 (*d*) rectangle 4

(*e*) the overall rectangle.

What can you conclude?



$(a + b)^2$ and $(a - b)^2$ are called *perfect squares.*

Notice that $(a + b)^2 = (a + b)(a + b)$

$= a^2 + ab + ab + b^2$

$= a^2 + 2ab + b^2$

Thus, we can state the perfect square expansion rule:

$$(a + b)^2 = a^2 + 2ab + b^2$$

We can remember the rule as follows:

Step 1: Square the *first term.*

Step 2: Add twice the product of the *first* and *last terms.*

Step 3: Add on the square of the *last term.*

Notice that $(a - b)^2 = (a + (-b))^2$

$= a^2 + 2a(-b) + (-b)^2$

$= a^2 - 2ab - b^2$

Once again, we have the square of the first term, twice the product of the first and last terms, and the square of the last term.

Example 26. *Expand and simplify:*

(*a*) $(x + 3)^2$ (*b*) $(x - 5)^2$

Solution. (*a*) $(x + 3)^2 = x^2 + 2 \times x \times 3 + 3^2 = x^2 + 6x + 9$

(*b*) $(x - 5)^2 = (x - 5)^2$

$= x^2 + 2 \times x \times (-5) + (-5)^2 = x^2 - 10x + 25.$

Example 27. *Expand and simplify:*

(*a*) $(2x^2 + 3)^2$ (*b*) $5 - (x + 2)^2$

Solution. (*a*) $(2x^2 + 3)^2 = (2x^2)^2 + 2 \times 2x^2 \times 3 + 3^2$

$= 4x^4 + 12x^2 + 9$

(*b*) $5 - (x + 2)^2 = 5 - [x^2 + 4x + 4]$

$= 5 - x^2 - 4x - 4 = 1 - x^2 - 4x.$

EXERCISE 3.8

1. Expand and simplify using the rule $(a + b)(a - b) = a^2 - b^2$:

(*a*) $(x + 2)(x - 2)$ (*b*) $(x - 3)(x + 3)$

2. Expand and simplify using the rule $(a + b)(a - b) = a^2 - b^2$:

(*a*) $(2x - 1)(2x + 1)$ (*b*) $(3x + 2)(3x - 2)$

(*c*) $(4y - 5)(4y + 5)$ (*d*) $(2y + 5)(2y - 5)$

3. Expand and simplify using the rule $(a + b)(a - b) = a^2 - b^2$:

(*a*) $(2a + b)(2a - b)$ (*b*) $(a - 2b)(a + 2b)$

(*c*) $(4x + y)(4x - y)$ (*d*) $(4x + 5y)(4x - 5y)$

4. (*a*) Using the difference of two squares expansion to show that:

(*i*) $43 \times 37 = 40^2 - 3^2$ (*ii*) $24 \times 26 = 25^2 - 1^2$

(*b*) Evaluate without using a calculator:

(*i*) 18×22 (*ii*) 49×51 (*iii*) 103×97.

5. Use the rule $(a + b)^2 = a^2 + 2ab + b^2$ to expand and simplify:

(*a*) $(x + 5)^2$ (*b*) $(x + 4)^2$ (*c*) $(x + 7)^2$

6. Expand and simplify:

(*a*) $(x - 3)^2$ (*b*) $(x - 2)^2$ (*c*) $(y - 8)^2$

7. Expand and simplify:

(*a*) $(3x + 4)^2$ (*b*) $(2a - 3)^2$ (*c*) $(3y + 1)^2$

8. Expand and simplify:

(*a*) $(x^2 + 2)^2$ (*b*) $(y^2 - 3)^2$ (*c*) $(3a^2 + 4)^2$

9. Expand and simplify:

(*a*) $3x + 1 - (x + 3)^2$ (*b*) $5x - 2 + (x - 2)^2$

(*c*) $(x + 2)(x - 2) + (x + 3)^2$ (*d*) $(x + 2)(x - 2) - (x + 3)^2$

3.11 FACTORISATION OF QUADRATIC EXPRESSIONS

Splitting the Middle Term

A *quadratic expression* is a quadratic trinomial of the form $ax^2 + bx + c$, where x is a variable and a, b, c are constants, $a \neq 0$.

Here, we will learn a useful technique for factorisation of a quadratic expression by *splitting the middle term.*

Consider $(2x + 3)(4x + 5)$

$$= 8x^2 + 10x + 12x + 15$$
$$= 8x^2 + 22x + 15$$

In reverse, $8x^2 + 22x + 15$

$$= \underbrace{8x^2 + 10x} + \underbrace{12x + 15}$$
$$= 2x(4x + 5) + 3(4x + 5)$$
$$= (4x + 5)(2x + 3)$$

So, we can factorise $8x^2 + 22x + 15$ into $(2x + 3)(4x + 5)$ by splitting the $+ 22x$ into a suitable sum, in this case $+ 10x + 12x$.

In general, if we start with a quadratic trinomial we will need a method to work out how to do the splitting.

Consider the expension in general detail:

$$(2x + 3)(4x + 5) = 2 \times 4 \times x^2 + [2 \times 5 + 3 \times 4]x + 3 \times 5$$
$$= 8x^2 + 22x + 15$$

The four numbers 2, 3, 4 and 5 are present in the *middle term,* and also in the *first* and *last* terms combined.

As 2 × 5 and 3 × 4 are factors of 2 × 3 × 4 × 5 = 120 this gives us the method for performing the splitting.

Step 1: Multiply the coefficient of x^2 and the constant term.

In our case, 8 × 15 = 120

Step 2: Look for the factors of this number which add to give the coefficient of the middle term.

What factors of 120 add to give us 22? The answer is 10 and 12.

Step 3: These numbers are the coefficients of the split terms.

So, the split is $10x + 12x$.

Example 28. *Factorise* $3x^2 + 17x + 10$.

Solution. For $3x^2 + 17x + 10$, 3 × 10 = 30

We need to find two factors of 30 which have a sum of 17. These are 2 and 15.

$$\therefore \quad 3x^2 + 17x + 10 = 3x^2 + 2x + 15x + 10$$
$$= x(3x + 2) + 5(3x + 2)$$
$$= (3x + 2)(x + 5).$$

Example 29. *Factorise:* $x^2 - 7x + 12$.

Solution. For $x^2 - 7x + 12$, 1 × 12 = 12

$$= x^2 - 4x - 3x + 12$$
$$= x(x - 4) - 3(x - 4)$$

We need to find two factors of 12 which have a sum – 7. These are – 3 and – 4.

$$\therefore \quad x^2 - 7x + 12 = (x - 3)(x - 4)$$

Remark. The sum is negative but the product is positive, so both numbers must be negative.

Example 30. *Factorise:* $6x^2 - 11x - 10$.

Solution. $6x^2 - 11x - 10$, $6 \times -10 = -60$

We need to find two factors of – 60 which have a sum of – 11.

These are – 15 and 4.

$$\therefore \quad 6x^2 - 11x - 10 = 6x^2 - 15x + 4x - 10$$
$$= 3x(2x - 5) + 2(2x - 5)$$
$$= (2x - 5)(3x + 2).$$

Remark. As the product is negative, the two numbers must be opposite in sign. Since the sum is negative, the larger number must be negative.

EXERCISE 3.9

1. Factorise:
(*a*) $x^2 + 4x + 3$ (*b*) $x^2 + 14x + 24$
(*c*) $x^2 + 10x + 21$ (*d*) $x^2 + 15x + 54$

2. Factorise:
(*a*) $x^2 - 3x + 2$ (*b*) $x^2 - 4x + 3$
(*c*) $x^2 - 5x + 6$ (*d*) $x^2 - 14x + 33$

3. Fully factorise:
(*a*) $2x^2 + 5x + 3$ (*b*) $2x^2 + 7x + 5$
(*c*) $7x^2 + 9x + 2$ (*d*) $3x^2 + 7x + 4$

4. Fully factorise:
(*a*) $2x^2 - 9x - 5$ (*b*) $3x^2 + 5x - 2$
(*c*) $3x^2 - 5x - 2$ (*d*) $2x^2 + 3x - 2$

5. Fully factorise:
(*a*) $15x^2 + 19x + 6$ (*b*) $15x^2 + x - 6$
(*c*) $15x^2 - x - 6$ (*d*) $30x^2 - 38x + 12$

REVIEW EXERCISE

1. Form algebraic expressions of the following statements.
(*a*) 10 more than twice a number.
(*b*) Product of x and y added to z.
(*c*) Difference of x and y divided by 5.
(*d*) Product of x and y added to their difference.
(*e*) Twice the number x added to square of it.

(*f*) Add 9 to the number x and divide the sum by 3.

(*g*) Subtract the number x from product of y and 3.

(*h*) Divide the difference of x and y by 2.

2. Find the value of algebraic expression by substituting the value of literals as given:

(*a*) $p^2q^2r^2 + 7$, if $p = 4$, $q = -1$, $r = 2$

(*b*) $x + y + z + xyz$, if $x = 9$, $y = -3$, $z = -1$

3. Add the algebraic expressions: $2x^3 - 9x^2 + 8$; $3x^2 - 6x - 5$; $7x^3 + 10x + 1$

4. Subtract the following expressions: $a^2 + ab + b^2$ from $4a^2 - 3ab + 2b^2$.

5. Multiply: $(-3ab^2) \times (3ab^2) \times (-2a^2b)$

6. Multiply: $\left(-\frac{1}{2}ab^2\right) \times (2a^2 - 3b^2)$

7. Multiply: $\left(3x + \frac{1}{2}y\right) \times \left(3x - \frac{1}{2}y\right)$

8. Multiply: $(a^3 - a^2b^2 + b^3) \times (a^2 - b^2)$

9. Simplify: $x(x + 4) + 3x(2x^2 - 1) + 4x^2 + 4$

10. Find the quotient and the remainder in the division of:

(*a*) $(a^3 - 5a^2 + 8a + 15)$ by $(a + 1)$

(*b*) $(5x^3 - 4c^2 + 2x - 3)$ by $(x^2 + 2x - 1)$

(*c*) $(a^4 + a^2 + 2)$ by $(a^2 - a + 1)$

11. Expand and simplify:

(*a*) $5(x - y)$ (*b*) $6(-x^2 + y^2)$ (*c*) $-2(x + 4)$

(*d*) $-3(2x - 1)$ (*e*) $x(x + 3)$ (*f*) $2x(x - 5)$

12. Expand and simplify:

(*a*) $2a(b - a) + 3a^2$ (*b*) $4x - 3x(x - 1)$ (*c*) $7x^2 - 5x(x + 2)$

13. Expand and simplify:

(*a*) $2(y - 3) - 4(2y + 1)$ (*b*) $3x - 4(2 - 3x)$

(*c*) $2(b - a) + 3(a + b)$ (*d*) $x(x + 4) + 2(x - 3)$

Simplify the following fractions. Write your answers in the lowest term.

14. $\frac{x}{a^2b} + \frac{y}{ab^2}$ **15.** $\frac{x+7}{3} + \frac{2x-3}{5}$

16. $\frac{2x}{x+1} + \frac{3}{(x+1)^2}$ **17.** $\frac{3}{(x+1)} - \frac{6}{(x+1)(x+3)}$

18. Copy and complete:

(*a*) $4x^2 - 8x = 4x(x - \ldots)$ (*b*) $2m + 8m^2 = 2m(\ldots + 4m)$

19. Copy and complete:

(*a*) $6a + 8ab = \ldots(3 + 4b)$ (*b*) $6x - 2x^2 = \ldots(3 - x)$

(*c*) $7ab - 7a = \ldots\ (b - 1)$ (*d*) $4ab - 6bc = \ldots(2a - 3c)$

20. Expand and simplify:

(*a*) $(x - 8)\ (x + 3)$ (*b*) $(2x + 1)\ (3x + 4)$

(*c*) $(1 - 2x)\ (4x + 1)$ (*d*) $(4 - x)\ (2x + 3)$

21. Expand and simplify:

(*a*) $(5a + 3)\ (5a - 3)$ (*b*) $(4 + 3a)\ (4 - 3a)$

22. Expand and simplify using the rule $(a + b)(a - b) = a^2 - b^2$:

(*a*) $(x + 1)\ (x - 1)$ (*b*) $(1 - x)\ (1 + x)$

23. Expand and simplify using the rule $(a + b)\ (a - b) = a^2 - b^2$:

(*a*) $(3x + 1)\ (3x - 1)$ (*b*) $(1 - 3x)\ (1 + 3x)$

24. Expand and simplify using the rule $(a + b)\ (a - b) = a^2 - b^2$:

(*a*) $(2x + 3y)\ (2x - 3y)$ (*b*) $(7x - 2y)\ (7x + 2y)$

25. Expand and simplify:

(*a*) $(3 - 2x)^2 - (x - 1)\ (x + 2)$ (*b*) $(1 - 3x)^2 + (x + 2)\ (x - 3)$

26. Factorise:

(*a*) $x^2 + 9x + 20$ (*b*) $x^2 + 8x + 15$

27. Factorise:

(*a*) $x^2 - 16x + 39$ (*b*) $x^2 - 19x + 48$

28. Fully factorise:

(*a*) $3x^2 + 13x + 4$ (*b*) $3x^2 + 8x + 4$

29. Fully factorise:

(*a*) $2x^2 + 3x - 5$ (*b*) $5x^2 - 14x - 3$

30. Fully factorise:

(*a*) $18x^2 - 12x + 2$ (*b*) $48x^2 + 72x + 27$

MULTIPLE CHOICE QUESTIONS (MCQs)

1. In the term $(-2pqr)$, numerical coefficient of p is:

(*a*) $-2qr$ (*b*) 2 (*c*) -2 (*d*) $2qr$

2. $14x^2y^2z$ and $7x^2zy^2$ are:

(*a*) unlike terms (*b*) like terms (*c*) binomial (*d*) None of these

3. If the length of a line is $2a + 3b$, if it is shortened by the measure $(a + b)$, find the remaining length of the line

(*a*) $3a + 4b$ (*b*) $a + 2b$ (*c*) $a - 2b$ (*d*) $a + b$

4. Daniel went to a park at a distance $x^2 + 2x + 3$ and came back on the same path. Find the total distance he travelled.
(*a*) $x^2 + 2x + 3$ (*b*) $2x^2 + 4x + 6$ (*c*) $2x^2 + 4x + 3$ (*d*) None of these

5. One side of a square plot is $(a + b + c)$. Find the perimeter.
(*a*) $2a + 2b + 2c$ (*b*) $4(a + b + c)$ (*c*) $4a + b + c$ (*d*) None of these

6. If we add $x^2 + 8x - 3$ and $-x^2 + 3 - 8x$, we get
(*a*) $2x^2 + 16x + 6$ (*b*) $-2x^2 - 16x - 6$ (*c*) 0 (*d*) 6

7. If $x = 3$, $y = -4$, $z = 2$, find the value of $x + y + z$.
(*a*) 9 (*b*) –9 (*c*) –1 (*d*) +1

8. Subtract the sum of $x^2 + 7x + 3$ and $3x^2 + 2x + 2$ from 1.
(*a*) $4x^2 + 9x + 4$ (*b*) $-4x^2 - 9x - 4$ (*c*) $-4x^2 - 9x - 5$ (*d*) None of these

9. $(x + 2)(x + 2) = ?$
(*a*) $(x^2 + 2x + 4)$ (*b*) $(x^2 + 4x + 4)$ (*c*) $(x^2 + 4)$ (*d*) $x^2 - 4$

10. $\left(\frac{1}{a}+\frac{1}{b}\right)\left(\frac{1}{a}-\frac{1}{b}\right) = ?$
(*a*) $\left(\frac{1}{a^2}-\frac{1}{b^2}\right)$ (*b*) $\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$ (*c*) $\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{b}\right)$ (*d*) $\frac{a^2-b^2}{ab}$

RECAP AT A GLANCE

- In algebra, a statement is basically sentences which are *either* true *or* false.
- A numerical statement is a *mathematical phrase* or *problem made up entirely of numerical characters.*
- In order to evaluate an algebraic expression, you must know the exact values for each variable.
- The product of two numbers with like signs is positive and the product of two numbers with unlike signs is negative.
- The quotient of two numbers with like signs is positive and the quotient of two numbers with unlike signs is negative.
- Algebraic fractions are fractions that contain at least one variable either in numerator or in denominator or in both.
- *Factorisation* is the process of writing an expression as a *product* of its *factors.*
- *Factorisation* is the reverse process of expansion.
- A *quadratic expression* is a quadratic trinomial of the form $ax^2 + bx + c$, where x is a variable and a, b, c are constants, $a \neq 0$.

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PERIOD-II

M10CH4

TOPIC 4

Number Base

4.1 THE BASE TEN SYSTEM

Numbers are usually written in base 10 (decimal numeral) but they can be written or converted to other bases as well. Base 10 is called the decimal numeral.

To indicate that a base other than 10 is being used, a small subscript is added after the number. Hence 32_4 indicates that the number has been written in base 4.

Let us consider the number (or numeral) 574. We know that it is read as 'five hundred seventy four'. It can be written as

$$
\begin{aligned}
574 &= 500 + 70 + 4 \\
&= 5 \times 100 + 7 \times 10 + 4 \times 1 \\
&= 5 \times 10^2 + 7 \times 10^1 + 4 \times 10^0
\end{aligned}
$$

The place values (or weights) of digits *from right to left* are powers of 10, i.e., 10^0, 10^1, 10^2. The pupils are familiar with this system from earlier days of schooling. This system is called the *decimal system.* In this system, we use the ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. We say that decimal system is a system with base ten. The base is written as a subscript. Thus, $574 = (574)_{10}$. Since the decimal system is the most commonly used, the base is usually not mentioned.

4.2 CONVERSION OF BASE 10 TO OTHER BASES

In fact, any integer greater than 1 can be taken as a base. In general, if the base is 'd', we take the digits 0, 1, 2, 3, ..., $d - 1$ to represent the number. The system with smallest possible base 2 is called the *binary system.* In this system, the only two digits used are 0 and 1. The number $(101)_2$ in the binary system is read as 'one zero one' and not as 'one hundred one'. We read the digits one by one. The system with base 3 is called the *ternary system*. In this system, the only three digits used are 0, 1 and 2.

To represent a number in the binary system (base 2), the place values are taken to be powers of the base, *i.e.*, 2^0, 2^1, 2^2, 2^3, ... from right to left.

To represent a number in the ternary system (base 3), the place values are taken to be powers of the base, *i.e.,* 3^0, 3^1, 3^2, 3^3, ... from right to left.

Continuing like this, to represent a number in the system with base '*d*', the only '*d*' digits used are 0, 1, 2, 3, ..., $d - 1$ and the place values are taken to be powers of *d*, i.e., d^0, d^1, d^2, d^3, ... from right to left.

Bases can be converted from other bases such as 2, 5, 8, 12 etc. to base 10 or vice versa.

In order to change from base 10 to a different base, a method involving *successive division* is used. The given decimal numeral is divided *repeatedly* by the appropriate base number, and the remainders, including zero, are noted at each stage. The division is continued until there is nothing left to divide. The answer is obtained by reading the remainders upwards as indicated by the arrows.

Example 1. *Convert 246_{10} to number base 8.*

Solution. Divide 246 repeatedly by 8.

8	246	
8	30	remainder 6 ↑
8	3	remainder 6 ↑
	0	remainder 3 ↑

∴ $246_{10} = 366_8$.

Example 2. *Convert 372_{10} to base eight.*

Solution. Divide 372 repeatedly by 8.

8	372	
8	46	remainder 4 ↑
8	5	remainder 6 ↑
	0	remainder 5 ↑

∴ $372_{10} = 564_8$.

Example 3. *Convert 1275_{10} to base twelve.*

Solution. Divide 1275 repeatedly by 12.

12	1275	
12	106	remainder 3
12	8	remainder 10
	0	remainder 8

∴ $1275_{12} = 8T3_{12}$.

Note: 10 is represented by T in base 12.

EXERCISE 4.1

1. Convert 7_{10} to number base 2 or binary numeral.
2. Convert 30_{10} to base four.
3. Convert 105 to number base two.

4.3 CONVERSION OF OTHER BASES TO BASE TEN

Converting from other bases to decimal numbers (i.e., base ten) is very simple, you remember that each digit in the other base number represents a power of that base number.

Example 4. *Convert 100101_2 to the corresponding base ten number.*

Solution. Write the digits in order, and count them from the Right to Left starting from zero:

1	0	0	1	0	1	: Digits
5	4	3	2	1	0	: Numbering

Use this listing to convert each digit to the power of two that it represents:

$1 \times 2^5 + 0 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$

$= 1 \times 32 + 0 \times 16 + 0 \times 8 + 1 \times 4 + 0 \times 2 + 1 \times 1$

$= 32 + 0 + 0 + 4 + 0 + 1 = 37$

Thus, 100101_2 converts to 37_{10}.

Example 5. *Convert 342_8 to number base 10.*

Solution. Write the digits in order, and count them from Right to Left starting from zero:

3	4	2	: Digits
2	1	0	: Numbering

Use this listing to convert each digit to the power of eight that it represents:

$$342_8 = (3 \times 8^2) + (4 \times 8^1) + (2 \times 8^0)$$
$$= (3 \times 64) + (4 \times 8) + (2 \times 1)$$
$$= 192 + 32 + 2 = 226_{10} = 226$$

Thus, 342_8 converts to 226_{10}.

Example 6. *Find in base ten the value of the 2 in 123_5.*

Solution.

$$123_5 = (1 \times 5^2) + (2 \times 5^1) + (3 \times 5^0)$$
$$= (1 \times 25) + (2 \times 5) + (3 \times 1)$$
$$= 25 + 10 + 3$$

$\therefore$ The value of the 2 in base ten is 10.

Note: **1.** A decimal numeral is the same as base 10.

2. Any number raised to the power zero is 1. *i.e.* $5^0 = 1$, $8^0 = 1$.

EXERCISE 4.2

1. Convert 1011_2 to number base 10 (decimal numeral).
2. Convert 2321_4 to number base 10 (or decimal numeral).
3. Convert 232_5 to decimal numeral.
4. Convert (*a*) 11.001_2 (*b*) 110.11_2 to a decimal numeral (or base 10).
5. Convert (*a*) 123_5 (*b*) 110011_2 to a decimal numeral (or base 10).

4.4 OPERATION ON NUMBERS INVOLVING NUMBER BASES OTHER THAN BASE TEN

Addition *and* Subtraction

Addition and subtraction involving number bases other than base ten is same as the numbers of base ten. But you need to be very careful about 'carry' during addition and 'borrow' during subtraction. It must be according to the number bases taken for operation.

In addition of numbers other than base ten, we add the two numbers and convert the resultant number in base we are working in and then forward 'carry' to the next column. Look at these examples.

Example 7. *Evaluate:*

(*a*) $123_5 + 241_5$ (*b*) $3047_8 + 442_8$

Solution.

(*a*)
$$\begin{array}{r} {\scriptsize 1\ 1\ \ } \\ 1\ 2\ 3_5 \\ +\ 2\ 4\ 2_5 \\ \hline 4\ 2\ 0_5 \end{array}$$

Explanation. Start from right ($3 + 2 = 5 = 10_5$). So, we write 0 and carry 1 to the next column.

The next column becomes ($1 + 2 + 4 = 7 = 12_5$). So, we write 2 and carry 1 to the next column.

The next column becomes $1 + 1 + 2 = 4$.

(*b*)
$$\begin{array}{r} {\scriptsize 1\ 1\ \ } \\ 3\ 0\ 4\ 7_8 \\ +\ 4\ 4\ 2_8 \\ \hline 3\ 5\ 1\ 1_8 \end{array}$$

Explanation. Start from right ($7 + 2 = 9 = 11_8$). So, we write 1 and carry 1 to the next column.

The next column becomes ($1 + 4 + 4 = 9 = 11_8$). So, we write 1 and carry 1 to the next column.

The next column becomes $1 + 4 = 5$.

So, we write 5 and 3 after it.

In subtraction of numbers other than base ten, we borrow the number equal to the base we are working in. Look at these examples.

Example 8. *Evaluate:*

(*a*) $324_5 - 233_5$ (*b*) $713_8 - 414_8$

Solution.

(*a*)
$$\begin{array}{r} 3\ 2\ 4_5 \\ -\ 2\ 3\ 3_5 \\ \hline 4\ 1_5 \end{array}$$

Explanation. Start from right ($4 - 3 = 1$). For the second column, we borrow 5 and add it to 2 to make 7 ($5 + 2 = 7$).

Now subtract 3 from 7 ($7 - 3 = 4$), write 4 in second column.

In next column left with 2 subtract 2 ($2 - 2 = 0$), write nothing in this column.

(*b*)
$$\begin{array}{r} 7\ 1\ 3_8 \\ -\ 4\ 1\ 4_8 \\ \hline 2\ 7\ 7_8 \end{array}$$

Explanation. Start from right, borrow 8 from the second column and add it to 3 then subtract 4 (*i.e.* $8 + 3 - 4 = 7_8$). Write 7 below in this column.

In next column again borrow 8 from next column and add it to 0 and subtract 1 *i.e.,* $8 + 0 - 1 = 7$. Write 7 below in this column. In next column $6 - 4 = 2$. Write 2 below in this column.

EXERCISE 4.3

1. Evaluate: $234_5 - 141_5$.

2. Perform the following operations with explanation in number base 8.

(*a*) 437 + 75 (*b*) 612 – 463.

3. Perform the following operations

(*a*) $321_5 - 123_5$ (*b*) $411_5 - 202_5$.

4.5 MULTIPLICATION

The table for binary multiplication is

×	0	1
0	0	0
1	0	1

Thus, $0 \times 0 = 0$, $0 \times 1 = 0$

$1 \times 0 = 0$, $1 \times 1 = 1$.

Example 9. *Perform* $(1101)_2 \times (110)_2$ *and check your answer.*

Solution.

$$\begin{array}{r} 1\ 1\ 0\ 1_2 \\ 1\ 1\ 0_2 \\ \hline 0\ 0\ 0\ 0 \\ 1\ 1\ 0\ 1\ \times \\ 1\ 1\ 0\ 1\ \times\ \times \\ \hline 1\ 0\ 0\ 1\ 1\ 1\ 0_2 \\ \hline \end{array}$$

Thus, $(1101)_2 \times (110)_2 = (1001110)_2$

Verification:

$$(1101)_2 = 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$$
$$= 8 + 4 + 0 + 1 = 13$$
$$(110)_2 = 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$$
$$= 4 + 2 + 0 = 6$$
$$(1001110)_2 = 1 \times 2^6 + 0 \times 2^5 + 0 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$$
$$= 64 + 0 + 0 + 8 + 4 + 2 + 0 = 78$$

and $13 \times 6 = 78.$

Hence the verification.

Example 10. *Perform $354_8 \times 463_8$ with explanation.*

Solution.

```
       3 5 4₈
       4 6 3₈
   ----------
     1 3 0 4
   2 6 1 0 ×
 1 6 6 0 × ×
 ------------
 2 1 5 4 0 4₈
```

Explanation:

(*i*) $4 \times 3 = 12$ [12 divided by 8, remainder 4 and carry over 1]

$3 \times 5 + 1 = 15 + 1 = 16$

[16 divided by 8, remainder 0 and carry over 2]

$3 \times 3 + 2 = 9 + 2 = 11$

[11 divided by 8, remainder 3 and carry over 1]

Then, we get $354_8 \times 3_8 = 1304_8$.

(*ii*) $6 \times 4 = 24$ [24 divided by 8, remainder 0 and carry over 3]

$6 \times 5 + 3 = 30 + 3 = 33$

[33 divided by 8, remainder 1 and carry over 4]

$6 \times 3 + 4 = 18 + 4 = 22$

[22 divided by 8, remainder 6 and carry over 2]

Then, we get $354_8 \times 6_8 = 2610_8$.

(*iii*) $4 \times 4 = 16$ [16 divided by 8, remainder 0 and carry over 2]

$4 \times 5 + 2 = 20 + 2 = 22$

[22 divided by 8, remainder 6 and carry over 2]

$4 \times 3 + 2 = 12 + 2 = 14$

[14 divided by 8, remainder 6 and carry over 1]

Then, we get $354_8 \times 4_8 = 1660_8$.

Add:

$$\begin{array}{r} {}^{1}\,1\,3\,0\,4_8 \\ {}^{1}\,2\,6\,1\,0\,0_8 \\ 1\,6\,6\,0\,0\,0_8 \\ \hline 2\,1\,5\,4\,0\,4_8 \\ \hline \end{array}$$

EXERCISE 4.4

1. Perform the following:

(*a*) $102_3 \times 21_3$ (*b*) $2102_3 \times 122_3$ (*c*) $(11_3)^2$ (*d*) $(20_3)^2$.

4.6 OPERATION IN OTHER BASES

The methods for adding and subtracting in other number bases are exactly the same as for decimal numerals.

Example 11. *Perform the following:* $111001_2 - 10101_2$.

Solution.

$$\begin{array}{r} 1\,1\,1\,0\,0\,1_2 \\ -\,1\,0\,1\,0\,1_2 \\ \hline 1\,0\,0\,1\,0\,0_2 \\ \hline \end{array}$$

Example 12. *If the addition performed in base 4, find the missing number.*

$$\begin{array}{r} 3\,1\,0\,1 \\ +\,*\,*\,*\,* \\ 2\,1\,3 \\ \hline 1\,0\,3\,0\,1 \\ \hline \end{array}$$

Solution. For the missing number, add 3101 and 213 in base 4 and subtract the result from 10301 in base 4.

$$\begin{array}{r} 3\,1\,0\,1_4 \\ +\,2\,1\,3_4 \\ \hline 3\,3\,2\,0_4 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{r} 1\,0\,3\,0\,1_4 \\ -\,3\,3\,2\,0_4 \\ \hline 3\,2\,1_4 \\ \hline \end{array}$$

Hence, the missing number is 321 in base 4.

EXERCISE 4.5

1. Perform the following: $11001_2 + 10111_2$.
2. Write the missing number if the following subtraction is in base 6.

$$\begin{array}{r} 4\,5\,2 \\ -\ *\,*\,* \\ \hline 2\,5\,4 \\ \hline \end{array}$$

3. Perform the following: $141_6 + 233_6 - 102_6$.

4.7 SIMPLE BASE EQUATIONS

To solve equations in which x occurs as the base of some numeral, reduce both sides to decimal system and then solve. Since x is a base, it must be a positive integer greater than 1.

Example 13. *Solve:* $(75)_x = (68)_{10}$.

Solution. Given $(75)_x = (68)_{10}$

Reducing the numeral on left side of the equation to base 10

$$(7 \times x^1 + 5 \times x^0)_{10} = (68)_{10}$$

$\Rightarrow \quad 7x + 5 = 68 \quad \Rightarrow \quad 7x = 68 - 5$

$\Rightarrow \quad 7x = 63 \quad \Rightarrow \quad x = 9$

Verification: $(75)_9 = 7 \times 9^1 + 5 \times 9^0$

$= 63 + 5 = 68.$

Example 14. *Solve:* $(132)_x = (42)_{10}$.

Solution. Given $(132)_x = (42)_{10}$

Reducing the numeral on left side of the equation to base 10

$$(1 \times x^2 + 3 \times x^1 + 2 \times x^0)_{10} = (42)_{10}$$

$\Rightarrow \quad x^2 + 3x + 2 = 42 \quad \Rightarrow \quad x^2 + 3x - 40 = 0$

$\Rightarrow \quad x^2 + 8x - 5x - 40 = 0 \quad \Rightarrow \quad x(x + 8) - 5(x + 8) = 0$

$\Rightarrow \quad (x + 8)(x - 5) = 0$

$\Rightarrow \quad x = -8, 5$

Since x is the base of a numerical, it cannot be negative.

Therefore, $x = 5$

Verification: $(132)_5 = 1 \times 5^2 + 3 \times 5^1 + 2 \times 5^0$

$= 25 + 15 + 2 = 42.$

Example 15. *Solve:* $(631)_x = (409)_{10}$.

Solution. Given $(631)_x = (409)_{10}$

Reducing the numeral on left side of the equation to base 10

$$(6 \times x^2 + 3 \times x^1 + 1 \times x^0)_{10} = (409)_{10}$$

$$\Rightarrow \quad 6x^2 \times 3x + 1 = 409$$

$$\Rightarrow \quad 6x^2 + 3x - 408 = 0$$

Dividing by 3

$$2x^2 + x - 136 = 0$$

$$\Rightarrow \quad 2x^2 + 17x - 16x - 136 = 0$$

$$\Rightarrow \quad x(2x + 17) - 8(2x + 17) = 0$$

$$\Rightarrow \quad (2x + 17)(x - 8) = 0$$

$$\Rightarrow \quad x = -\frac{17}{2}, 8$$

Since x is the base of a numeral, it cannot be negative.

Therefore, $x = 8$

Verification:

$$(631)_8 = 6 \times 8^2 + 3 \times 8^1 + 1 \times 8^0$$
$$= 384 + 24 + 1 = 409.$$

EXERCISE 4.6

Solve the following equations for x:

1. $(31)_x = (16)_{10}$

2. $(46)_x = (38)_{10}$

3. $(101)_x = (5)_{10}$

4. $(201)_x = (19)_{10}$

5. $(245)_x = (101)_{10}$

6. $(396)_x = (468)_{10}$

REVIEW EXERCISE

1. Convert 72_{10} to base five.

2. Convert 77 to number base two.

3. Convert 12.04_5.

4. Convert

(*a*) 1204_5 (*b*) 11001_2

5. Mr. Henry had 3540_8 oranges. He sold 402_8 and 317_8 got spoilt. How many oranges were left? Leave your answer in number base ten.

6. Perform the following:

(*a*) $21_3 \times 21_3$ (*b*) $1322_8 \times 13_8$

7. Evaluate: $311_4 + 213_4 - 332_4$ leaving your answer in base 10.

8. If 263 + 441 = 714, find the number base that has been used.

Solve the following equations for x:

9. $(105)_x = (54)_{10}$ **10.** $(123)_x = (38)_{10}$

MULTIPLE CHOICE QUESTIONS (MCQs)

1. Convert 104_{10} to a binary numeral.
(*a*) 1101000 (*b*) 1101100 (*c*) 1011010 (*d*) 1010100

2. Write 1101101_2 in base ten.
(*a*) 125 (*b*) 31 (*c*) 145 (*d*) 109

3. Convert 11001_2 to a decimal numeral.
(*a*) 7 (*b*) 6 (*c*) 14 (*d*) 25

4. Convert 206 to base five numeral.
(*a*) 3321_5 (*b*) 411_5 (*c*) 4011_5 (*d*) 1311_5

5. Express 87_{10} as a base five numeral.
(*a*) 3202_5 (*b*) 322_5 (*c*) 302_5 (*d*) 3022_5

6. Convert 134_5 to base ten numeral.
(*a*) 220 (*b*) 16 (*c*) 44 (*d*) 40

7. Express 57_{10} as a base two (binary) numeral.
(*a*) 100111_2 (*b*) 101011_2 (*c*) 11010_2 (*d*) 111001_2

8. Convert 320_5 to a base ten numeral.
(*a*) 25 (*b*) 77 (*c*) 85 (*d*) 86

RECAP AT A GLANCE

- Base 10 is called the decimal numeral.
- The numeration system in base two is referred to as the binary system.
- The system with base 3 is called the *ternary* system.
- 2 is the smallest possible base.
- Bases can be converted from other bases such as 2, 5, 8 etc. to base 10 or from base 10 to bases 2, 5, 8 etc.
- A decimal numeral is the same as base 10.
- Any number raised to the power zero is 1.
- To solve an equation, first change the base on both sides of the equation to base ten.

❑ ❑ ❑

PERIOD-III

M10CH5

TOPIC 5

Plane Geometry

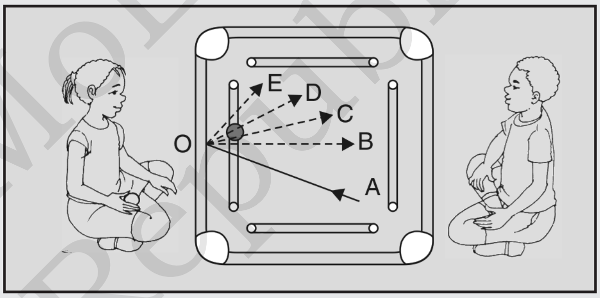
5.1 MEASURING *AND* DRAWING ANGLES

ACTIVITY 1

The following figure shows some of the angles a striker can make on a carrom board.

Name all the angles shown in this figure.

You should get 10 angles. Five of these are:



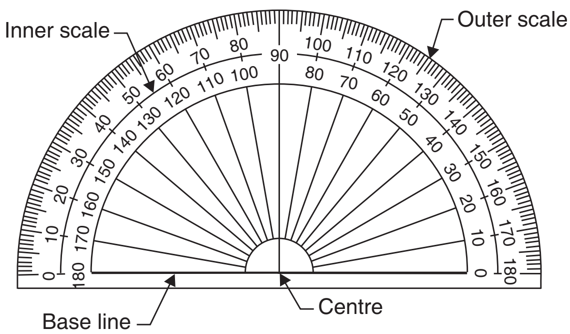
∠ AOB, ∠ AOC, ∠ AOD, ∠ AOE, ∠ BOC.

Write the other five angles.

By this activity, we observe that angles are formed when two lines meet.

Introduction to the Various Parts of the Protractor

If you look at the protractor carefully, you will see that there are two sets of measurements written on it, *i.e.,* divisions marked in opposite directions. These are called *scales*. There is an *inner scale* and an *outer scale*, both having 0° to 180° in different directions.

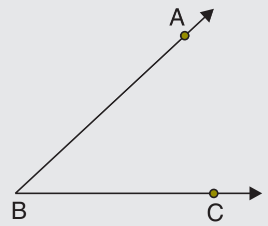


You can use a protractor to measure angles.

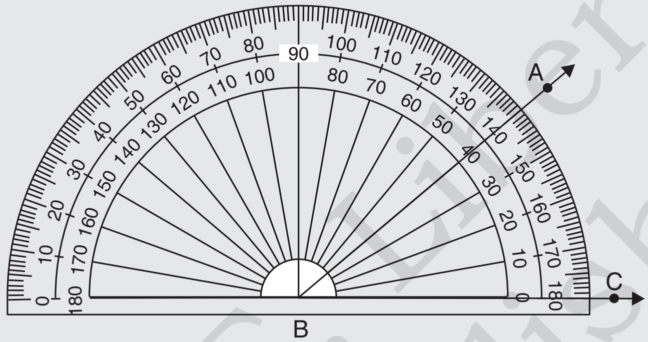
ACTIVITY 2

Measuring Angles Using the Protractor

Suppose you want to measure an angle ABC.



Given ∠ ABC



Measuring ∠ ABC

Step 1: Place the protractor so that the centre point of the baseline lies on the vertex B of the angle.

Step 2: Adjust the protractor (without shifting the centre from the vertex) so that one arm BC of the angle is along the baseline.

Step 3: There are two '*scales*' on the protractor: Read that scale which has the 0° mark coinciding with the baseline (inner scale in this example)

Step 4: The mark shown by BA on the curved edge gives the degree measure of the angle, *i.e.*, read the measure of this angle where the other arm BA crosses the scale. We write $m \angle ABC = 40°$, or simply $\angle ABC = 40°$.

Note: The length of the arms does not affect the measure of the angle.

ACTIVITY 3

Drawing Angles Using the Protractor

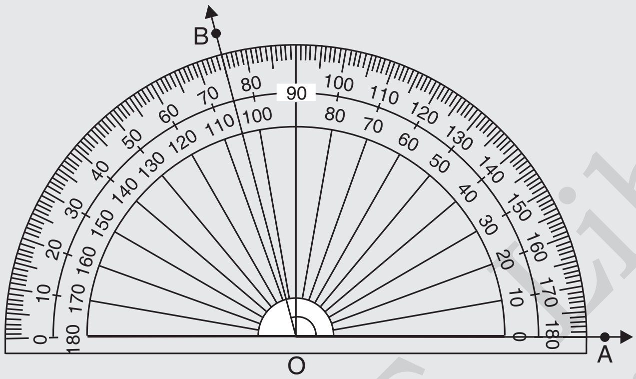
You can draw angles using the protractor.

Suppose you want to draw an angle of 105° using the protractor.

Step 1: Draw a ray OA.

Step 2: Place the protractor in such a way that its centre lies exactly at O and the base line lies along OA.

Step 3: Starting from 0° on the side of A, move the eyes and look for the 105° mark on the protractor. Mark a point B against this 105° mark.



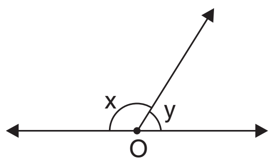
Step 4: Remove the protractor and draw the ray OB. Then, ∠AOB is the required angle whose measure is 105°.

Angles on a Straight Line

Draw a straight line to a point O on a line as shown in the following figure.

Now, measure the two angles formed using the protractor.

$$x = 122°, \quad y = 58°$$



Let us add their results,

$$x + y = 122° + 58° = 180°$$

Thus, $x + y = 180°$.

The sum of angles on a straight line is 180°, *i.e.,* the angles are supplementary. A pair of supplementary angles form a *linear pair* (180°) when placed adjacent to each other.

Hence, if the sum of two or more adjacent angles is 180°, then the non-common arms of the angles form a straight angle.

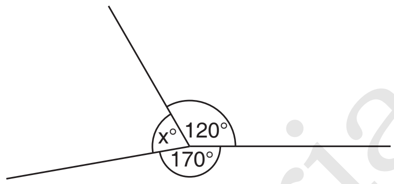
Note: Angles measuring 180° are called straight angles.

5.2 CALCULATING ANGLES

Angles at a Point and on a Line

One complete revolution is equivalent to a rotation of 360° about a point. Similarly, half a complete revolution is equivalent to a rotation of 180° about a point. These facts can be seen clearly by looking at *either* a circular angle measurer *or* a semi-circular protractor.

Example 1. *Calculate the size of the angle x in the adjacent figure.*



Solution. The sum of all the angles around a point is 360°.

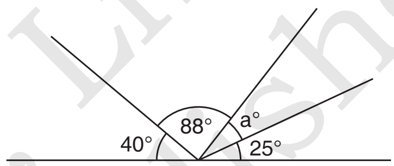
$\therefore \quad 120° + 170° + x = 360°$

$\Rightarrow \quad x = 360° - 120° - 170°$

$x = 70°$

Therefore angle x is 70°.

Example 2. *Calculate the size of angle a in the adjacent diagram.*



Solution. The sum of all the angles at a point on a straight line is 180°.

$\therefore \quad 40° + 88° + a + 25° = 180°$

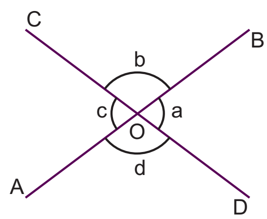
$\Rightarrow \quad a = 180° - 40° - 88° - 25°$

$a = 27°$

Therefore angle a is 27°.

Vertically Opposite Angles

When two straight lines intersect each other, they form four angles. A pair of angles with no common arm are called vertically opposite angles (*abbreviated as vert. opp. ∠s*).



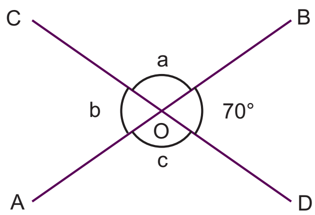
In the figure, a and c are vertically opposite angles and so are angles b and d.

If two straight lines intersect, then the vertically opposite angles are equal.

Thus, $\angle a = \angle c$ and $\angle b = \angle d$.

Example 3. *In the figure, two straight lines AB and CD intersect at O. Find the angles a, b and c.*

Solution. Angles COB and DOB form a linear pair.



Therefore,

$\angle COB + \angle DOB = 180°$

$\Rightarrow \quad a + 70° = 180°$

$\Rightarrow \quad a = 180° - 70° = 110°$

As $\angle AOC = \angle BOD$ (vert. opp. $\angle s$)

$\Rightarrow$ $b = 70°$

$\angle AOD = \angle COB$ (vert. opp. $\angle s$)

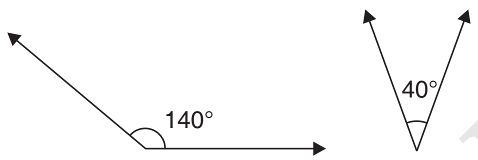
$\Rightarrow$ $c = a = 110°$

Hence, $a = 110°$, $b = 70°$ and $c = 110°$.

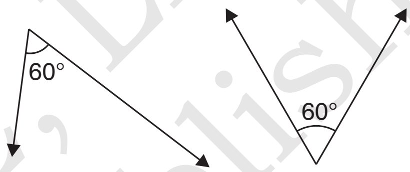
EXERCISE 5.1

1. Check which of the following pairs of angles form a linear pair:

(*a*)

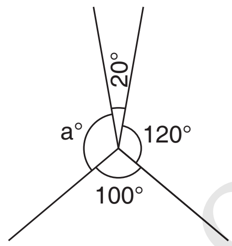


(*b*)

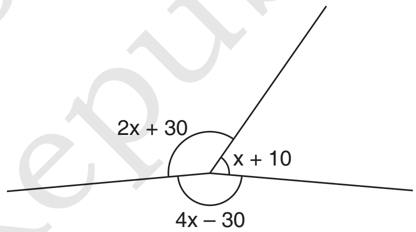


2. Find the value of the lettered angles in the following diagrams.

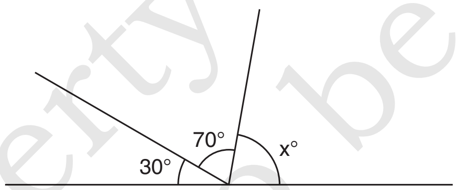
(*a*)



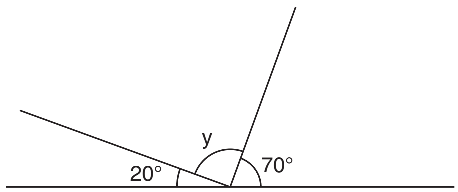
(*b*)



3. Find the value of x in the figure below.



Q. 3



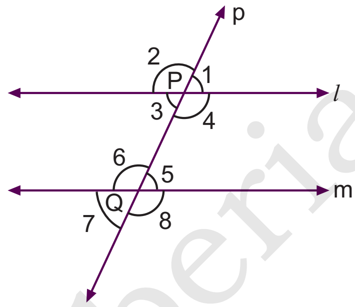
Q. 4

4. Find the value of y in the above figure.

5.3 ANGLE PROPERTIES OF PARALLEL LINES

A line that intersects two or more lines (not necessarily parallel lines) at *distinct* points is called a *transversal.* A transversal to two parallel lines is of special significance in the study of geometry.

If two parallel lines l and m are cut by a transversal p at points P and Q respectively, then eight angles marked 1 to 8 are formed as shown in the adjacent figure. These angles have special names, individually as well as in pairs. These pairs of angles have special properties. If any one of these eight angles is known, then the remaining seven can be easily obtained.



Now we state some important properties of these angles. If two parallel lines are cut by a transversal, then

(*i*) pairs of corresponding angles are equal,

i.e., $\angle 1 = \angle 5$, $\angle 2 = \angle 6$, $\angle 3 = \angle 7$ and $\angle 4 = \angle 8$

(*ii*) pairs of alternate interior angles are equal,

i.e., $\angle 3 = \angle 5$ and $\angle 4 = \angle 6$

(*iii*) pairs of alternate exterior angles are equal

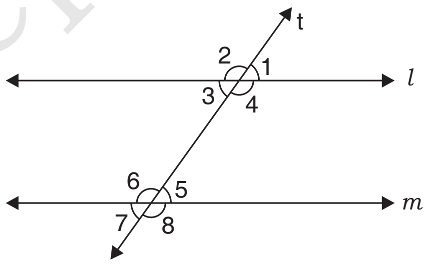
i.e., $\angle 1 = \angle 7$ and $\angle 2 = \angle 8$

(*iv*) the sum of interior angles on the same side of the transversal is 180°,

i.e., $\angle 3 + \angle 6 = 180°$

and $\angle 4 + \angle 5 = 180°$

Example 4. *In the following diagram line $l \parallel m$ and t is a transversal. If $\angle 1 = 30°$, find all the angles from 2 to 8.*



Solution. Given $\angle 1 = 30°$.

(*i*) $\angle 1$ and $\angle 5$ are corresponding angles.

$\therefore$ $\angle 5 = \angle 1 \Rightarrow \angle 5 = 30°$ [$\because \angle 1 = 30°$]

(*ii*) $\angle 1 + \angle 2 = 180°$ [Linear pair]

$\Rightarrow$ $\angle 2 = 180° - \angle 1 \Rightarrow 180° - 30° = 150°$

[$\because \angle 1 = 30°$]

(*iii*) $\angle 2 = \angle 4$ [Vertically opposite angles]

$\Rightarrow$ $\angle 4 = 150°$ [$\because \angle 2 = 150°$]

(*iv*) $\angle 1 = \angle 3$ [Vertically opposite angles]

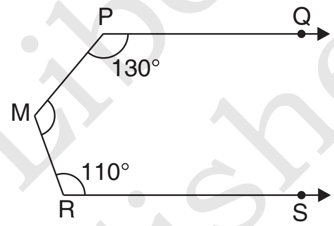
$\Rightarrow$ $\angle 3 = 30°$ [$\because \angle 1 = 30°$]

(*v*) $\angle 4 = \angle 6$ [Alternate interior angles]

$\Rightarrow$ $\angle 6 = 150°$ [$\because \angle 4 = 150°$]

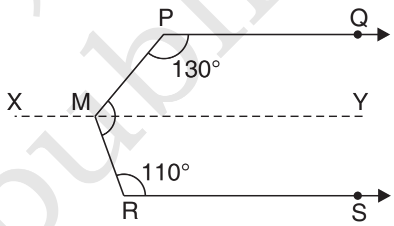
(*vi*)	$\angle 3 = \angle 7$	[Corresponding angles]
$\Rightarrow$	$\angle 7 = 30°$	[$\because \quad \angle 3 = 30°$]
(*vii*)	$\angle 4 = \angle 8$	[Corresponding angles]
$\Rightarrow$	$\angle 8 = 150°$	[$\because \quad \angle 4 = 150°$]
Thus,	$\angle 1 = \angle 3 = \angle 5 = \angle 7 = 30°$	
	$\angle 2 = \angle 4 = \angle 6 = \angle 8 = 150°$.	

Example 5. *In the given diagram PQ ∥ RS, ∠QPM = 130° and ∠SRM = 110°. Find ∠PMR.*



Solution.

Construction. Through M, draw a line XY ∥ PQ, and PM is a transversal.



$\therefore \quad \angle QPM + \angle PMY = 180°$

[Sum of interior angles on the same side of a transversal is 180°]

$\Rightarrow \quad \angle PMY = 180° - 130°$ [$\because \quad \angle QPM = 130°$]

$= 50°$

Now, XY ∥ PQ and PQ ∥ RS $\Rightarrow$ XY ∥ RS

XY ∥ RS and MR is a transversal.

$\because \quad \angle SRM + \angle RMY = 180°$

[Sum of interior angles on the same side of a transversal is 180°]

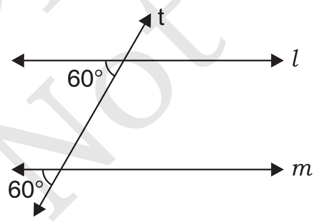
$\Rightarrow \quad \angle RMY = 180° - 110° = 70°$ [$\because \quad \angle SRM = 110°$]

$\Rightarrow \quad \angle PMR = \angle PMY + \angle RMY = 50° + 70° = 120°$

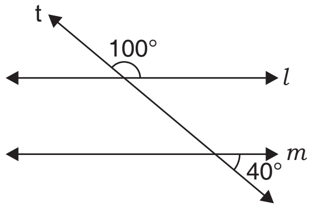
Hence, $\angle PMR = 120°$.

EXERCISE 5.2

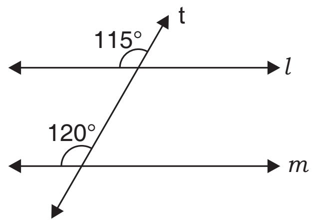
1. In the figures given below, find out whether line m is parallel to line l.



(a)

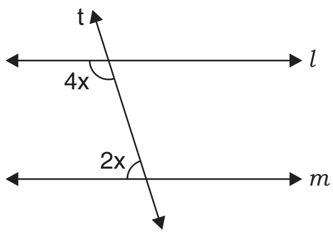


(b)

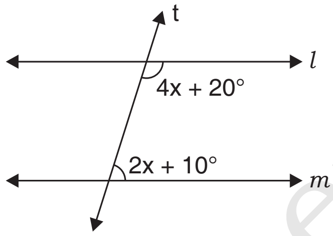


(c)

2. If line *l* is parallel to line *m*, find the value of *x*.

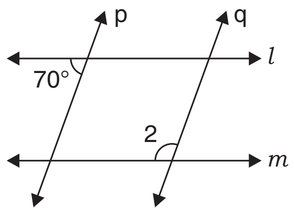


(a)

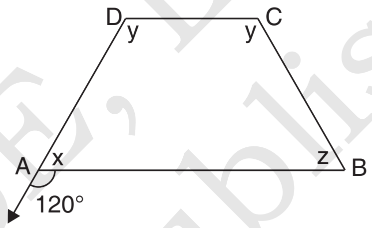


(b)

3. (*a*) In the diagram given below $l \parallel m$, $p \parallel q$ and $\angle 1 = 70°$. Find $\angle 2$.
(*b*) In the diagram given below AB || CD. Find the angles *x*, *y*, *z*.



(a)



(b)

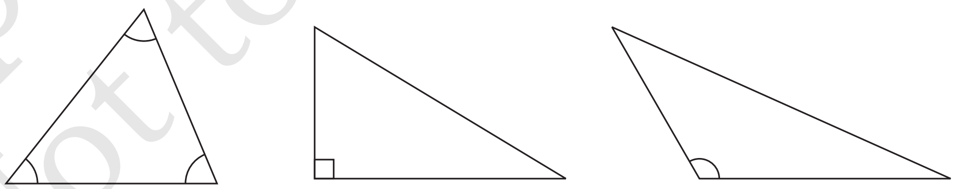
5.4 TRIANGLES

Name of Triangles

Triangles can be described in terms of their sides or their angles, or both.

(a) Naming Triangles Based on Angles

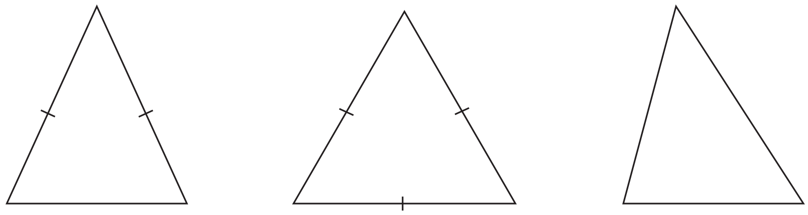
- An *acute-angled* triangle has *all its angles less than 90°.*
- A *right-angled* triangle has an angle of 90°.
- An *obtuse-angled* triangle has one angle greater than 90°.



Acute-angled　　Right-angled　　Obtuse-angled

(b) Naming Triangles Based on Sides

- An *isosceles* triangle has two sides of equal length, and the angles opposite the equal sides are equal.



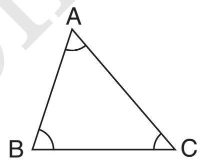
Isosceles triangle Equilateral triangle Scalene triangle

- An *equilateral* triangle has three sides of equal length and three equal angles.
- A *scalene* triangle has three sides of different lengths and all three angles are different.

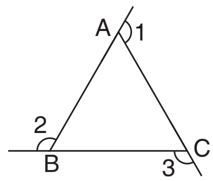
5.5 ANGLE PROPERTIES OF TRIANGLES

1. The sum of three interior angles of a triangle is always 180°. This property is called the *angle sum property* of a triangle. Therefore,

$$\angle A + \angle B + \angle C = 180°.$$

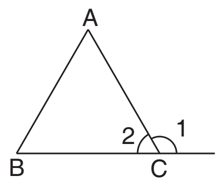


2. The exterior angles of a triangle always add up to 360°. Therefore $\angle 1 + \angle 2 + \angle 3 = 180°$.



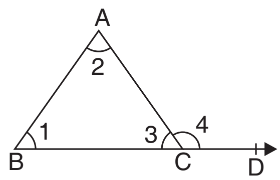
3. The sum of consecutive interior and exterior angle is supplementary. Therefore,

$$\angle 1 + \angle 2 = 180°.$$



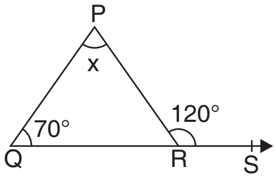
4. If one side of a triangle is extended, then the exterior angle so formed is equal to the sum of the two interior opposite angles. This property is called an *exterior angle property* of a triangle.

Therefore, $\angle 4 = \angle 1 + \angle 2.$

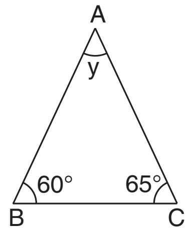


Example 6. *In the following diagrams, find the unknown angles:*

(*i*)



(*ii*)



Solution.

(*i*) In ΔPQR,

$\angle P + \angle Q = \angle PRS$ [Exterior angle property of a triangle]

∴ $x + 70° = 120° \Rightarrow x = 120° - 70° = 50°$

Thus, $\angle QPR = 50°$

(*ii*) In ΔABC,

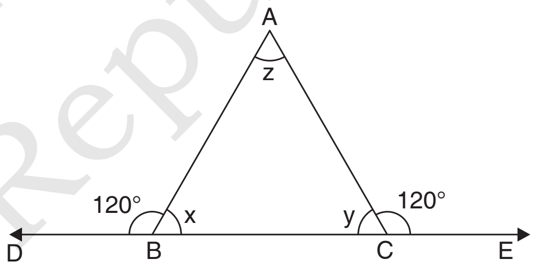
$\angle A + \angle B + \angle C = 180°$ [Sum of the angles of a triangle is 180°]

$\Rightarrow y + 60° + 65° = 180°$

$\Rightarrow y = 180° - 125° = 55°$

Thus, $\angle BAC = 55°$

Example 7. *In the given ΔABC, exterior angles ∠ACE = 120° and ∠ABD = 120°. Find y and z and identify the type of triangle in the given diagram.*



Solution. $120° + x = 180°$ [Linear pair at point B]

$\Rightarrow x = 180° - 120° = 60°$ ...(1)

$120° + y = 180°$ [Linear pair at point C]

$\Rightarrow y = 180° - 120° = 60°$ ...(2)

In ΔABC,

$x + y + z = 180°$ [Angle sum property of a Δ]

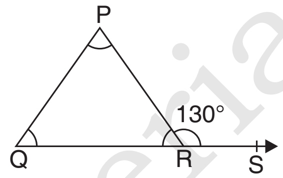
$\Rightarrow 60° + 60° + z = 180°$

[Substituting the values from (1) and (2)]

$\Rightarrow z = 180° - 120° = 160°$

Since, all the angles of the triangle are 60° each, it is an equivalent triangle.

Example 8. *In ΔPQR, ∠P and ∠Q are in the ratio 2 : 3. Find all the angles of the triangle.*



Solution. Let the angles ∠P and ∠Q be $2x$ and $3x$ respectively.

$$\angle PRQ + \angle PRS = 180°$$

[Linear pair at the point R]

$$\Rightarrow \quad \angle PRQ + 130° = 180°$$

$$\Rightarrow \quad \angle PRQ = 180° - 130° = 50°$$

In ΔPQR,

$$\angle P + \angle Q + \angle R = 180° \qquad \text{[Angle sum property of a } \Delta]$$

$$\Rightarrow \quad 2x + 3x + 50° = 180°$$

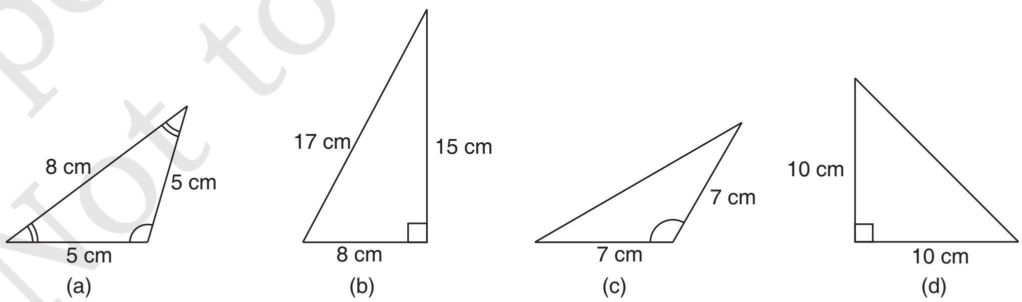
$$\Rightarrow \quad 5x = 180° - 50° = 130° \quad \Rightarrow \quad x = \frac{130°}{5} = 26°$$

$\therefore \quad 2x = 52°$ and $3x = 78$

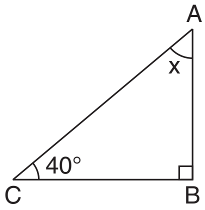
Thus, ∠P = 52°, ∠Q = 78° and ∠R = 50°.

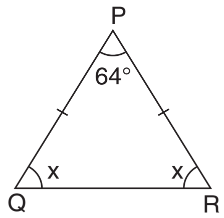
EXERCISE 5.3

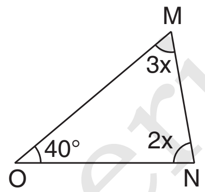
1. Name the types of following triangles:
   (*a*) Triangle with lengths of sides 7 cm, 8 cm and 9 cm.
   (*b*) ΔABC with AB = 8.7 cm, AC = 7 cm and BC = 6 cm.
   (*c*) ΔPQR such that PQ = QR = PR = 5 cm.
   (*d*) ΔDEF with ∠D = 90°
2. Name each of the following triangles in two different ways: (you may judge the nature of the angle by observation)



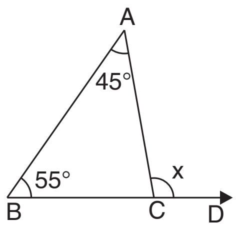
3. Find the value of *x*, *y*, *z* in the following triangles:

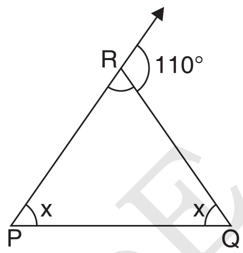
(*a*)


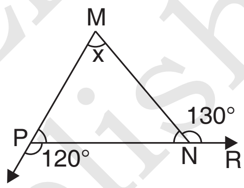
(*b*)


(*c*)


4. Find the value of *x* in the following triangles:

(*a*)


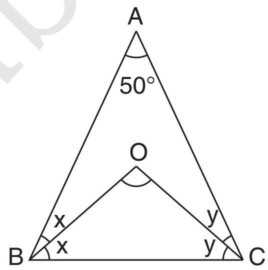
(*b*)


(*c*)


5. In the given diagram, find ∠BOC.

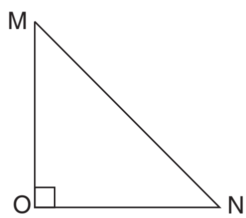
6. An exterior angle of a triangle is 120°. If interior opposite angles are in the ratio of 3 : 5, find the angles of the triangle.

7. If the angles of a triangle are in the ratio 2 : 3 : 4, find the angles.



Q. 5

5.6 RIGHT ANGLED TRIANGLES



- One angle is always 90° or right angle.
- The side opposite angle of 90° is the hypotenuse.
- The hypotenuse is always the longest side.
- The sum of the other two interior angles is equal to 90°.
- The other two sides adjacent to the right angle are called base and perpendicular.

5.7 SQUARES *AND* SQUARE ROOTS

Squares

If a number is multiplied by itself, the product obtained is called the *square* of that number.

For a given number 'x', the square of x is $(x \times x) = x^2$ we read it as x square.

For example: $5^2 = (5 \times 5) = 25$

We say that the square of 5 is 25.

Example 9. *Find the square of 36.*

Solution. Square of 36 = 36^2

$= 36 \times 36 = 1296$

$$\begin{array}{r} 36 \\ \times\ 36 \\ \hline 216 \\ 1080 \\ \hline 1296 \end{array}$$

Perfect Square Number

The numbers 1, 4, 9, 16, etc. can respectively be expressed as (1^2), (2^2), (3^2), (4^2), etc.

Thus 1, 4, 9, 16, etc. are known as square numbers or perfect squares.

In general, a natural number is called a square number or *perfect square* if it is the square of a natural number.

Example 10. *Is 1764 a perfect square? If so, find the number whose square is 1764.*

Solution. Resolving 1764 into prime factors, we get

$1764 = (2 \times 2 \times 3 \times 3 \times 7 \times 7)$

$= (2 \times 2) \times (3 \times 3) \times (7 \times 7)$

[Grouping the factors in pairs of identical factors]

$= 2^2 \times 3^2 \times 7^2$

$= (2 \times 3 \times 7)^2 = (42)^2$

2	1764
2	882
3	441
3	147
7	49
7	7
	1

Hence, 1764 is a perfect square as it can be expressed as the product of pairs of equal factors.

So, 42 is the number whose square is 1764.

Square Roots

We know that 16, 64, 81 are perfect suqares.

$$16 = 4^2 = 4 \times 4$$
$$64 = 8^2 = 8 \times 8$$
$$81 = 9^2 = 9 \times 9$$

The square roots of 16, 64, 81 are 4, 8, 9 respectively.

Thus, square root of a number x is that number, which when multiplied by itself, gives x as the product.

The square root is a number x is denoted by $\sqrt[2]{x}$ or only $\sqrt{x}$.

$\sqrt{}$ symbol is known as radical sign.

Thus, $\sqrt{16} = 4$, $\sqrt{64} = 8$, $\sqrt{81} = 9$.

Square Root of a Perfect Square by Prime-Factorisation Method

To find the square root of a perfect square number we follow the given steps:

Step 1: Resolve the given number into prime factors.

Step 2: Make pairs of identical prime factors.

Step 3: Take one prime factor from every pair and find the product of these factors.

The product gives us the square root of the given number.

Example 11. *Find the square root of (i) 676 and (ii) 3025 by the method of prime factorisation.*

Solution. (*i*) By prime factorisation, we get

2	676
2	338
13	169
13	13
	1

$$676 = (2 \times 2) \times (13 \times 13)$$

$$\therefore \quad \sqrt{676} = 2 \times 13 = 26.$$

(*ii*) Resolving 3025 into prime factors, we get

5	3025
5	605
11	121
11	11
	1

$$3025 = (5 \times 5) \times (11 \times 11)$$

$$\therefore \quad \sqrt{3025} = 5 \times 11 = 55.$$

EXERCISE 5.4

1. Is 343 a perfect square?
2. Show that each of the following is a perfect square. Also find the number whose perfect square is the given number.

   (*a*) 625 (*b*) 7056 (*c*) 1521

3. By what least number should the given number be multiplied to get a perfect square number? In each case, find the number whose square is the new number.

(*a*) 252 (*b*) 4851 (*c*) 1452

4. By prime factorisation method, find the square root of these numbers.

(*a*) 36 (*b*) 64 (*c*) 900 (*d*) 576

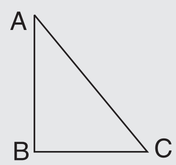
5.8 PYTHAGORAS THEOREM

Pythagoras was a Greek philosopher who lived from 580 BC to 500 BC.

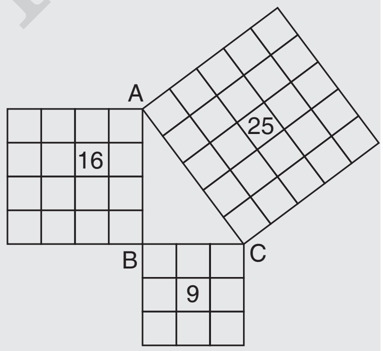
He gave a wonderful relation between the lengths of the sides of a right angle triangle. This relationship is now known as Pythagoras theorem.

ACTIVITY 4

1. Take a white chart paper and draw a right triangle with sides AB = 4 cm, BC = 3 cm and AC = 5 cm as shown in the figure.



2. Now, using a pair of scissors, cut three squares of dimensions 3 cm, 4 cm and 5 cm from a grid sheet and paste them along the sides of the triangle as shown in the figure.



3. Count the number of squares on each side.

Number of squares on side AB = 16 = 4^2

Number of squares on side BC = 9 = 3^2

Number of squares on side AC = 25 = 5^2

Clearly, $16 + 9 = 25$

$$AB^2 + BC^2 = AC^2$$

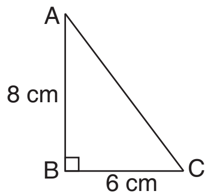
According to Pythagoras theorem in a right angle triangle, the square of the hypotenuse equals to the sum of the squares of its two remaining sides.

Converse of Pythagoras Theorem

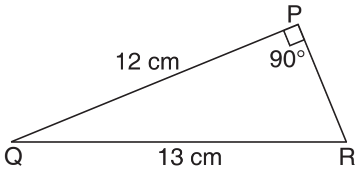
In a triangle, if sum of the squares of two sides is equal to the square of the hypotenuse, then the triangle is a right-angled triangle.

Example 13. *Find the third side of the following right-angled triangles.*

(*i*)



(*ii*)



Solution.

(*i*) In ΔABC, $\angle B$ is a right angle.

$\therefore \quad (AB)^2 + (BC)^2 = (AC)^2$ [According to Pythagoras theorem]

$\Rightarrow \quad (8)^2 + (6)^2 = (AC)^2 \quad \Rightarrow \quad 64 + 36 = (AC)^2$

$\Rightarrow \quad 100 = AC^2 \quad \Rightarrow \quad AC = \sqrt{100} = 10$

Side $\quad AC = 10$ cm.

(*ii*) In ΔPQR, $\angle P$ is a right angle.

$\therefore \quad (PQ)^2 + (PR)^2 = (QR)^2$ [According to Pythagoras theorem]

$\Rightarrow \quad (12)^2 + (PR)^2 = (13)^2 \quad \Rightarrow \quad 144 + (PR)^2 = 169$

$\Rightarrow \quad (PR)^2 = 169 - 144 \quad \Rightarrow \quad (PR)^2 = 25$

$\Rightarrow \quad PR = \sqrt{25} = 5$

Side $\quad PR = 5$ cm.

5.9 PYTHAGOREAN TRIPLES

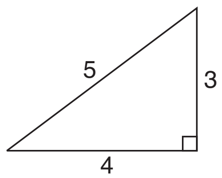
The simplest right angled triangle with sides of *integer* length is the *3-4-5 triangle.*

The numbers 3, 4 and 5 satisfy the rule $3^2 + 4^2 = 5^2$

The set of positive integers $\{a, b, c\}$ is a *Pythagorean triple* if it obeys the rule $a^2 + b^2 = c^2$.

Other examples are:

$\{5, 12, 13\}, \{7, 24, 25\}, \{8, 15, 17\}$.



Example 1. *Show that* $\{5, 12, 13\}$ *is a Pythagorean triple.*

Solution. We find the square of the *largest* number first

$13^2 = 169$

and $\quad 5^2 + 12^2 = 25 + 144 = 169$

$\therefore \quad 5^2 + 12^2 = 13^2$

So, $\{5, 12, 13\}$ is a Pythagorean triple.

Example 2. *Find k if {9, k, 15} is a Pythagorean triple.*

Solution. By Pythagoras theorem,

$$9^2 + k^2 = 15^2 \quad \text{[By Pythagoras theorem]}$$

$$\Rightarrow \quad 81 + k^2 = 225$$

$$\Rightarrow \quad k^2 = 144 \Rightarrow k = \sqrt{144} \quad \{\text{as } k > 0\}$$

$$\Rightarrow \quad k = 12.$$

EXERCISE 5.5

1. In the following right angle triangles, height (a) and base (b) are given. Find the hypotenuse (c).

(a) a = 6 cm b = 8 cm (b) a = 1.5 cm b = 2 cm

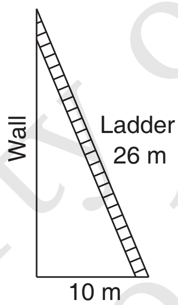
(c) a = 40 cm b = 9 cm (d) a = 8 cm b = 15 cm

2. Sides of the following triangles are given. Find out whether they are right angled triangles:

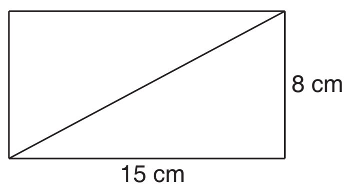
(a) 4 cm, 5 cm, 6 cm (b) 15 cm, 20 cm, 25 cm

(c) 6 cm, 8 cm, 11 cm (d) 5 cm, 12 cm, 13 cm

3. A ladder that is 26 m long, rests against a vertical wall, with its foot 10 m away from the wall. How high up the wall will the ladder reach?



Q. 3



Q. 4

4. The sides of a rectangle are 15 cm and 8 cm. Find the length of diagonal.

5. Which of the following are Pythagorean triples?

(a) {8, 15, 17} (b) {6, 8, 10} (c) {5, 6, 7} (d) {14, 48, 50}

6. Find k if the following are Pythagorean triples:

(a) {8, 15, k} (b) {k, 24, 26} (c) {14, k, 50}

5.10 PARALLELOGRAMS *AND* TRAPEZIUM–KITES, RHOMBUSES

Since quadrilaterals have special names. Each special name has a special *property* related to it.

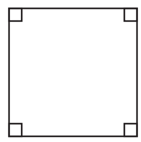
Note: You can often use these properties to find *lengths* and *angles* in quadrilaterals. Quadrilaterals can be sorted out into two main sets: those which are parallelograms and those which are not.

Parallelograms	Non-parallelograms
Parallelograms, rhombus, square and rectangle	Trapezium, kites and irregular quadrilaterals

Square

A square has:

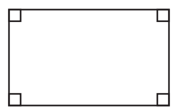
- all angles equal to 90°
- all sides equal
- opposite sides parallel
- diagonals equal in length and bisect each other
- diagonals cross at right angles
- diagonals bisect corner angles



Rectangle

A rectangle has:

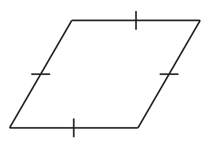
- all angles equal to 90°
- opposite sides equal
- opposite sides parallel
- diagonals equal in length and bisect each other



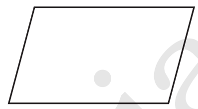
Rhombus

A rhombus has:

- all sides equal
- opposite sides parallel
- opposite angles equal
- diagonals bisect each other
- diagonals cross at right angles
- diagonals bisect corner angles



Parallelogram

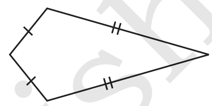


A parallelogram has:

- opposite angles equal
- opposite sides equal
- opposite sides parallel
- diagonals bisect each other

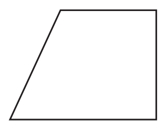
Kite

A kite has:



- one pair of opposite angles equal
- two pairs of adjacent sides equal
- diagonals cross at right angles
- only one diagonal is bisected
- only one pair of opposite angles is bisected

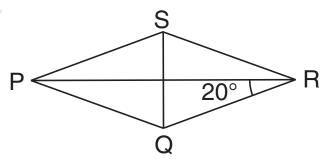
Trapezium



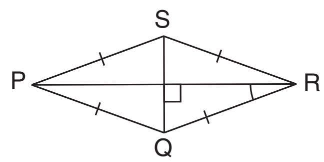
A trapezium has:

- one pair of opposite sides parallel

Example 14. *In the given figure, PQRS is a rhombus and ∠PRQ = 20°. What is the value of ∠QSR?*



Solution. A rhombus has all sides equal and the diagonals bisect each other at right angles.



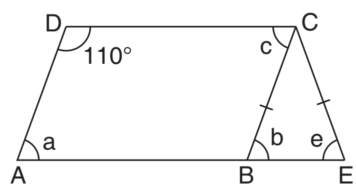
Therefore ∠RQS = 90° – 20° = 70°.

Triangle ∠QRS is isosceles,

Therefore ∠QSR = ∠RQS = 70°

(Base angles of an isosceles triangle are equal)

Example 15. *Find the size of each lettered angle in the given diagram.*



Solution. ABCD is a prallelogram and BCE is an isosceles triangle.

Opposite angles of a parallelogram are equal.

i.e., ∠ABC = 110°

Adjacent angles on a straight line add up to 180°.

i.e., $b + \angle ABC = 180°$

$\Rightarrow$ $b = 180° - 110° = 70°$

Base angles of an isosceles triangle are equal.

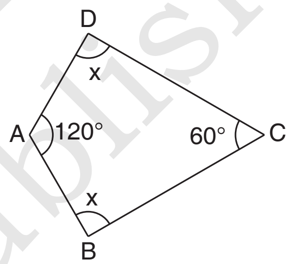
i.e., $e = b = 70°$

Alternate angles are equal. *i.e.*, $c = b = 70°$

Opposite angles of a parallelogram are equal,

i.e., $a = c = 70°$

Example 16. *Find the value of x in the given figure.*



Solution. As we know that the sum of all angles of a quadrilateral (kite) is 360°. Therefore,

$120° + x + 60° + x = 360°$

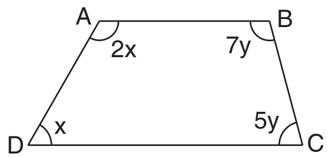
$\Rightarrow$ $2x + 180° = 360°$

$\Rightarrow$ $2x = 360° - 180° = 180°$

$\Rightarrow$ $x = \frac{180°}{2} = 90°.$

Example 17. *ABCD is a trapezium such that AB || CD as shown in the adjacent figure:*

Find the angles of the trapezium.



Solution. In trapezium, the angles on the same side between the pair of parallel sides are supplementary *i.e.*, they are add up to 180°.

$\therefore$ $\angle A + \angle D = 180°$

$\Rightarrow$ $2x + x = 180°$

$\Rightarrow$ $3x = 180°$ $\Rightarrow$ $x = \frac{180°}{3} = 60°$

Similarly, $\angle B + \angle C = 180°$ $\Rightarrow$ $7y + 5y = 180°$

$\Rightarrow$ $12y = 180°$ $\Rightarrow$ $y = \frac{180°}{12} = 15°$

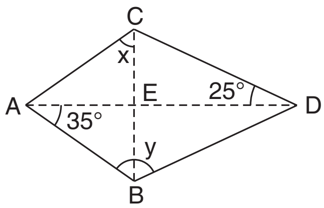
$\therefore$ Angles of the trapezium are:

$\angle A = 2x = 2 \times 60° = 120°$; $\angle B = 7y = 7 \times 15° = 105°$

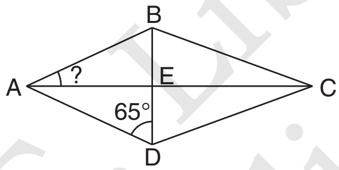
$\angle C = 5y = 5 \times 15° = 75°$; $\angle D = x = 60°.$

EXERCISE 5.6

1. In a parallelogram PQRS, if ∠QRS = $2x$, ∠PQR = $4x$ and ∠PSQ = $4x$, find the angles of the parallelogram.
2. ABCD is a rhombus with ∠ABC = 50°. Determine ∠ACD.
3. Two consecutive angles of a prallelogram are $(x + 60)°$ and $(2x + 30)°$. What special name can you give to this parallelogram?
4. Find the values of x and y in the given figure.

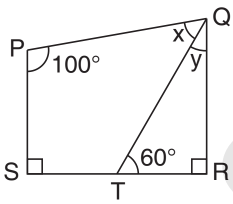


Q. 4

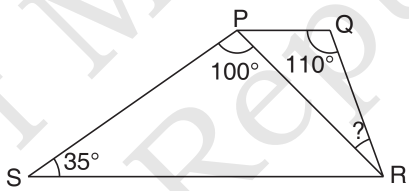


Q. 5

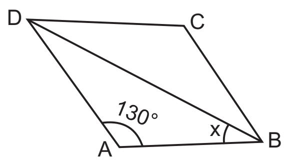
5. If ∠ADE = 65°, what is ∠BAE?
6. Given figure is a trapezium in which PS || QR. Find the values of x and y.



Q. 6



Q. 7



Q. 8

7. PQRS is a trapezium in which PQ || SR. Find the unknown angle ∠QRP.
8. In the above figure, ABCD is a rhombus. Find the value of x.

5.11 POLYGONS

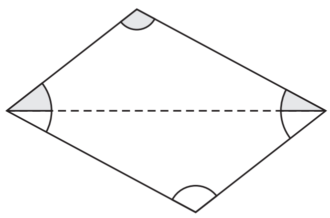
Any closed figure/curve made up of only line segments is called a polygon. Triangle, quadrilateral, pentagon etc. are examples of polygons.

The Interior Angles of a Polygon

Now consider finding the sum of angles of a quadrilateral.

We can construct a diagonal of the quadrilateral as shown.

The constructed diagonal divide the quadrilateral into two triangles.



We have already seen that the sum of interior angles of a triangle is 180°.

So, the sum of the interior angles of a quadrilateral

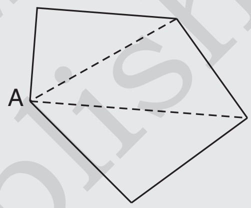
$= 2 \times$ Sum of interior angles of a triangle

$= 2 \times 180° = 360°$

We can generalise this process to find the sum of the interior angles of any polygon.

ACTIVITY 5

1. Draw any pantagon (5-sided polygon) and label one of its vertices A. Draw in all the diagonals from A.



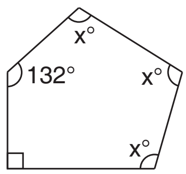
2. Repeat 1 for a hexagon, a heptagon (7-gon), an octagon, and so on, drawing diagonals from one vertex only.
3. Copy and complete the following table:

Polygon	Number of sides	Number of diagonals from A	Number of triangles	Angle sum of polygon
Quadrilateral	4	1	2	$2 \times 180° = 360°$
Pentagon				
Hexagon				
Octagon				
20-gon				

We have discovered that:

The sum of the interior angles of any n-sided polygon is $(n - 2) \times 180°$.

Example 18. *Find x, giving a brief reason.*



Solution. The pentagon has 5 sides.

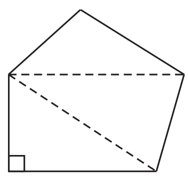
∴ Sum of interior angles = $(5 - 2 = 3) \times 180° = 540°$

i.e., $x + x + x + 132° + 90° = 540°$

$\Rightarrow 3x + 222° = 540°$

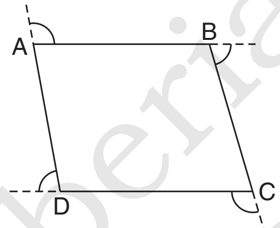
$\Rightarrow 3x = 318°$

$\Rightarrow x = 106°$.



The Exterior Angles of a Polygon

The *exterior angles* of a polygon are formed by extending the sides in either direction.



The shaded angle is said to be an exterior angle of quadrilateral ABCD at vertices A, B, C and D.

The sum of the exterior angles of any polygon is always 360°.

Example 19. *A regular polygon has 15 sides. Calculate the size of each interior angle.*

Solution. For a 15-sided polygon, each, exterior angle is 360° ÷ 15 = 24°

$\therefore$ Each interior angle is 180° – 24° = 156°.

Note: A regular polygon has equal interior angles and equal sides.

EXERCISE 5.7

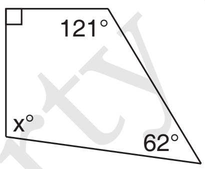
1. Find the sum of the interior angles of:

(*a*) a quadrilateral (*b*) a pentagon

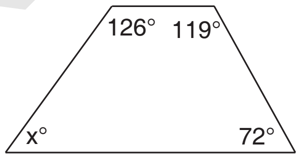
(*c*) a hexagon (*d*) an octagon

2. Find the value of the unknown in:

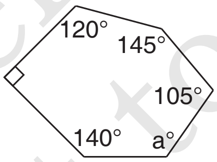
(*a*)



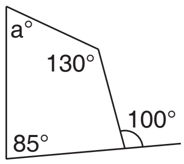
(*b*)



(*c*)

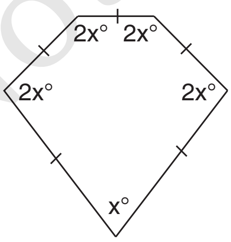


(*d*)

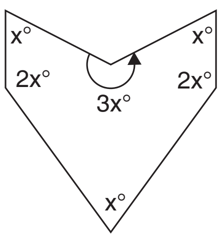


3. Find the value of *x* in each of the following, giving a reason.

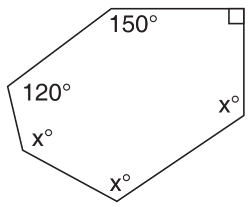
(*a*)

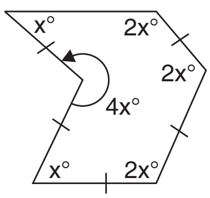


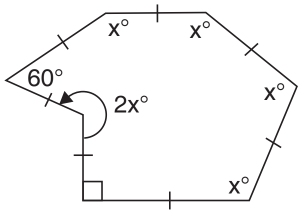
(*b*)

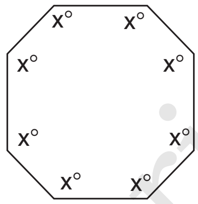


(*c*)

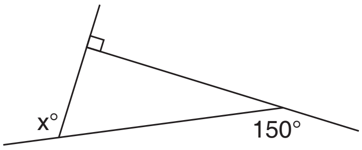


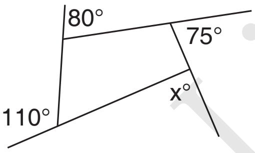
(*d*) 

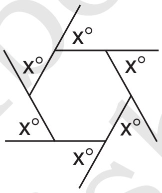
(*e*) 

(*f*) 

4. Solve for x:

(*a*) 

(*b*) 

(*c*) 

5. Calculate the size of each interior angle of these regular polygons:

(*a*) with 5 sides (*b*) with 8 sides

6. Calculate the number of sides of a regular polygon given that an exterior angle is:

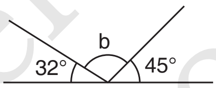
(*a*) 45° (*b*) 15° (*c*) $\frac{1}{2}^{\circ}$

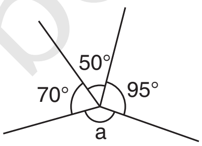
7. Calculate the number of sides of a regular polygon with an interior angle of:

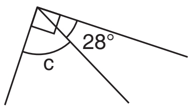
(*a*) 120° (*b*) 150° (*c*) 179°

REVIEW EXERCISE

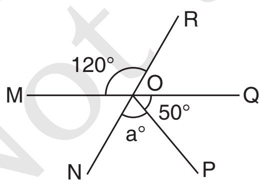
1. Calculate the size of each of the lettered angles in the diagrams below.

(*a*) 

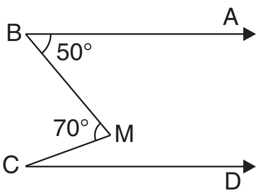
(*b*) 

(*c*) 

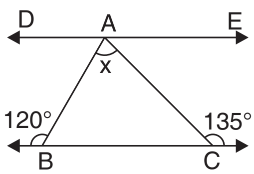
2. Find the value of a in the figure below.



Q. 2



Q. 3



Q. 4

3. In the above diagram, AB || CD, ∠ABM = 50°, ∠BMC = 70°. Find ∠MCD.

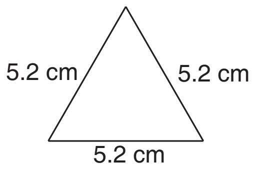
4. In the above diagram, DE || BC. Find the value of angle x.

5. Name the types of following triangles:

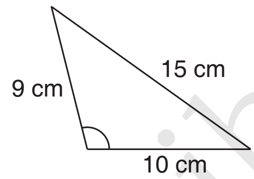
(*a*) ΔXYZ with $\angle Y = 90°$ and XY = YZ.

(*b*) ΔLMN with $\angle L = 30°$, $\angle M = 70°$ and $\angle N = 80°$.

6. Name each of the following triangles in two different ways: (you may judge the nature of the angle by observation)



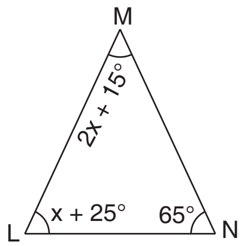
(a)



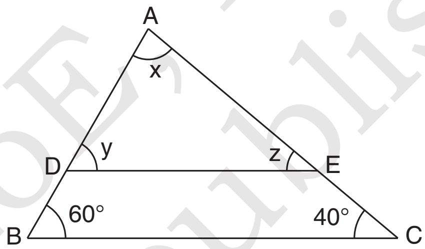
(b)

7. Find the value of x, y, z in the following triangles:

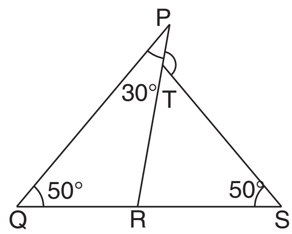
(*a*)



(*b*)



8. In the given diagram, find $\angle PTS$ and $\angle PRS$.



Q. 8

9. Show that each of the following is a perfect square. Also find the number whose perfect square is the given number.

(*a*) 324 (*b*) 4489

10. By Prime Factorisation method, find the square root of these numbers.

(*a*) 5184 (*b*) 7056

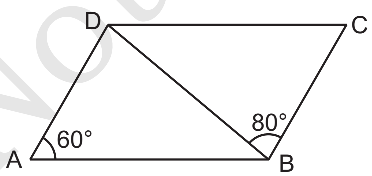
11. Which of the following are Pythagorean triples?

(*a*) {1, 2, 3} (*b*) {20, 48, 52} (*c*) {9, 12, 15} (*d*) {12, 16, 18}

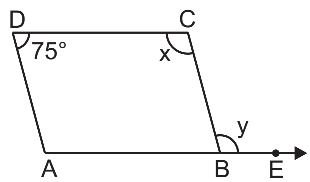
12. Find k if the following are Pythagorean triples:

(*a*) {15, 20, k} (*b*) {k, 45, 51} (*c*) {11, k, 61}

13. In parallelogram ABCD in the figure, $\angle DAB = 60°$ and $\angle DBC = 80°$. Find, $\angle ABD$.



Q. 13

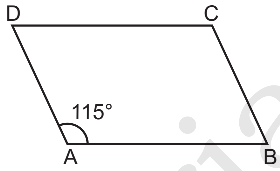


Q. 14

14. ABCD is a parallelogram in which $\angle ADC = 75°$ and side AB is produced to point E as shown in the figure. Find $(x + y)$.

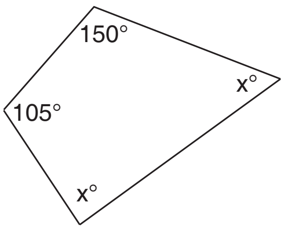
15. Two opposite angles of a parallelogram are $(3x - 2)°$ and $(50 - x)°$. Find the measure of each angle of the parallelogram.

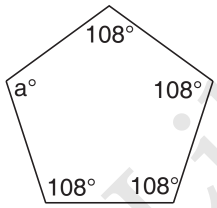
16. In a ||gm ABCD, if ∠A = 115°, find ∠B, ∠C and ∠D.



Q. 16

17. Find the value of the unknown in:

(*a*)


(*b*)


18. Calculate the size of each interior angle of these regular polygons:
(*a*) with 10 sides (*b*) with 20 sides
(*c*) with 100 sides (*d*) with n sides

19. Calculate the number of sides of a regular polygon given that an exterior angle is 2°.

20. Calculate the number of sides of a regular polygon with an interior angle of 175°.

MULTIPLE CHOICE QUESTIONS (MCQs)

1. At 3 o'clock, the angle between hour and minute hand is:
(*a*) 30° (*b*) 60° (*c*) 90° (*d*) 180°

2. An angle equal to its supplementary angle is:
(*a*) 45° (*b*) 90° (*c*) 180° (*d*) None of these

3. If angles $4x$ and $2x$ form a linear pair, then x is:
(*a*) 15° (*b*) 120° (*c*) 60° (*d*) 30°

4. If a transversal intersects two parallel lines, then:
(*a*) corresponding angles are equal
(*b*) alternate interior angles are equal
(*c*) co-interior angles are supplementary
(*d*) all of the above

5. Two adjacent angles:
(*a*) have a common vertex (*b*) have a common arm
(*c*) do not overlap (*d*) all of the above

6. Angles $(5x - 10°)$ and $(2x - 20°)$ are co-interior angles when a transversal intersects two parallel lines. Their measures are:
(*a*) 150°, 30° (*b*) 140°, 40° (*c*) 100°, 80° (*d*) 120°, 60°

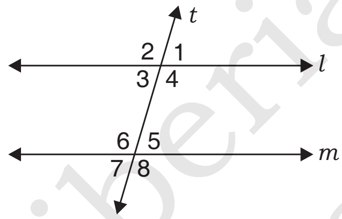
7. Two parallel lines are intersected by a transversal, so co-interior angles are $(3x + 20°)$ and x. Find the value of x.

(*a*) 60° (*b*) 120° (*c*) 140° (*d*) 40°

8. In the given diagram, lines l and m are non-parallel. Line t is a transversal cutting lines l and m at two distinct points. Which of the following holds true?

(*a*) $\angle 1 = \angle 3$ (*b*) $\angle 4 = \angle 6$

(*c*) $\angle 1 = \angle 7$ (*d*) None of these



Q. 8

9. Triangle having one angle 120° is called:

(*a*) Acute-angled Δ (*b*) Right-angled Δ

(*c*) Obtuse-angled Δ (*d*) None of these

10. If sides of the triangle are 4.2 cm, 3.5 cm and 4.2 cm, such Δ is called:

(*a*) Scalene (*b*) Equilateral

(*c*) Isosceles (*d*) Obtuse-angled

11. If the angles of a Δ are $(2x + 5°)$, $(3x + 10°)$, $(x + 15°)$, then $x =$

(*a*) 25° (*b*) 30° (*c*) 90° (*d*) 180°

12. If an exterior angle of a triangle is 120° and one of the opposite interior angles is 60°, then another opposite interior angle is equal to:

(*a*) 60° (*b*) 80° (*c*) 120° (*d*) 40°

RECAP AT A GLANCE

- Angles are formed when two lines meet.
- The sum of angles on a straight line is 180°.
- A line that intersects two or more lines (not necessarily parallel lines) at *distinct* points is called a *transversal.*
- An *acute-angled* triangle has *all its angles less than 90°.*
- A *right-angled* triangle has an angle of 90°.
- An *obtuse-angled* triangle has one angle greater than 90°.
- The sum of three interior angles of a triangle is always 180°.
- The exterior angles of a triangle always add up to 360°.
- The sum of consecutive interior and exterior angle is supplementary.
- The hypotenuse is always the longest side.
- The other two sides adjacent to the right angle are called base and perpendicular.
- Any closed figure/curve made up of only line segments is called a polygon.

❑ ❑ ❑

SEMESTER-II

PERIOD-IV

M10CH6

TOPIC

6

Linear Equations and Inequalities

6.1 EQUALITY *AND* EQUIVALENCE

Formation of Linear Equations

In Semester-I, you have learnt the formation of algebraic expression under given conditions. Let us expand our study to form linear equations by using concept of equality.

Example 1. *Make an equation for each of the given conditions:*

(*i*) *Sum of a number x and 5 is 25.*

(*ii*) *Take away 3 from a number y gives twice the number.*

(*iii*) *x multiplied by 5 gives 1.*

(*iv*) *The product of –2 and a number p is equal to 0.*

(*v*) *The quotient of y divided by 4 is –10.*

Solution.

(*i*) Sum of a number x and 5 is $x + 5$. But this sum is equal to 25.

$\therefore \quad x + 5 = 25$

(*ii*) Take away 3 from a number y is $y - 3$.

Twice the number y is $2y$.

These two results are equal.

$\therefore \quad y - 3 = 2y$

(*iii*) x multiplied by 5 is $5 \times x$ or $5x$.

This product gives 1.

$\therefore \quad 5x = 1.$

(*iv*) The product of –2 and p is $-2p$.

This is equal to 0.

$\therefore \qquad -2p = 0$

(*v*) The quotient of y divided by 4 is $\frac{y}{4}$.

This equals to –10.

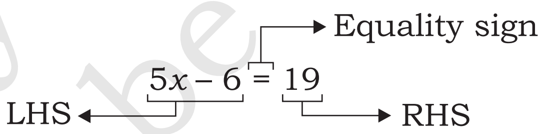
$\therefore \qquad \frac{y}{4} = -10.$

Other examples of linear equations in one variable are:

Linear Equations in one variable	Variable
$3x - 2 = 4$	x
$x + 12 = 2x - 3$	x
$16y = y + 15$	y
$y - 4 = 4$	y
$-8z + 2 = -10$	z

A linear equation is an equation which contains the variable which is not raised to any power other than 1.

Every equation consits of left hand side (LHS), an equality sign (=) and right hand side (RHS). Look at this example:



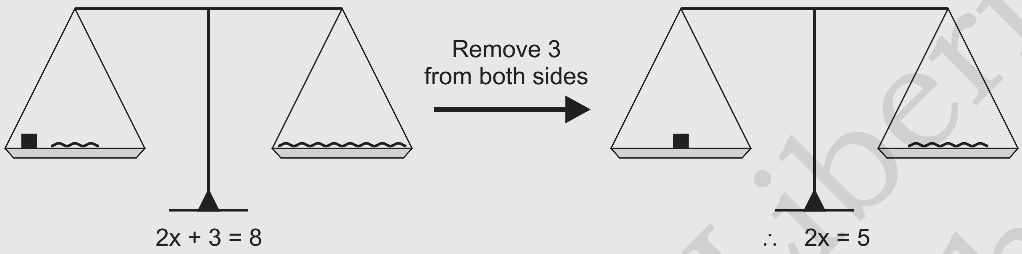
Here, LHS is $5x - 6$ and RHS is 19. In every equation LHS and RHS are always equal.

Properties of Equalities and Equivalence

Two equations that have the same solution are called *equivalent* equations *e.g.* 4 + 3 = 2 + 5. And this as we learned in the above sub-section *i.e.*, formation of linear equations is shown by the equality sign =. An inverse operation are two operations that undo each other *e.g.* addition and subtraction or multiplication and division. You can perform the same inverse operation on each side of an equivalent equation without changing the equality.

ACTIVITY 1

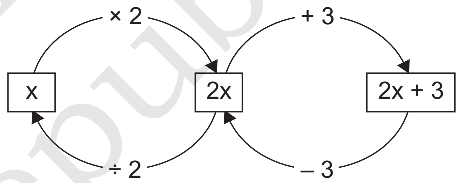
Compare the balance of weights:



We perform operations on both sides of each equation in order to *isolate the unknown.*

We consider how the expression has been *built up* and then *isolate the unknown* by using *inverse operations* in *reverse order.*

Consider the equation $2x + 3 = 8$, the LHS is built up by starting with x, multiplying by 2, then adding 3.



So, to isolate x, we first subtract 3 from both sides, then divide both sides by 2.

Example 2. *Solve for x:* $3x + 7 = 22$.

Solution. Here, $3x + 7 = 22$

$\therefore\quad 3x + 7 - 7 = 22 - 7$ (Subtracting 7 from both sides)

$\therefore\quad 3x = 15$ (Simplifying)

$\therefore\quad \dfrac{3x}{3} = \dfrac{15}{3}$ (Dividing both sides by 3)

$\therefore\quad x = 5$ (Simplifying)

Check: LHS $= 3 \times 5 + 7 = 22$ $\therefore$ LHS = RHS.

EXERCISE 6.1

1. Solve for x:

(*a*) $x + 11 = 0$ (*b*) $4x = -12$ (*c*) $5x + 35 = 0$ (*d*) $4x - 5 = -17$

2. Solve for x:

(*a*) $8 - x = -3$ (*b*) $-4x = 22$ (*c*) $3 - 2x = 11$ (*d*) $6 - 4x = -8$

3. Solve for x:

(*a*) $\frac{x}{4} = 7$ (*b*) $\frac{2x}{5} = -6$ (*c*) $\frac{x}{2} + 3 = -5$ (*d*) $\frac{x}{4} - 2 = -5$

4. Solve for x:

(*a*) $\frac{2x+11}{3} = 0$ (*b*) $\frac{1}{2}(3x+1) = -4$

(*c*) $\frac{1+2x}{5} = 7$ (*d*) $\frac{1-2x}{5} = 3.$

6.2 FINDING THE SOLUTION SET OF A LINEAR EQUATION

Solving an equation is the process of finding the solutions of the equation. The set of all solutions of an equation is called the *solution set* of the equation.

For example: $x = 3$ is a solution of the equation $4x - 9 = 3$ because, substituting $x = 3$ in the equation, we have $4 \times 3 - 9 = 3$ or $12 - 9 = 3$ or $3 = 3$ which is true.

Therefore, the solution set of the equation $4x - 9 = 3$ is $S = \{x : x = 3\}$ or $\{3\}$.

The solution set of a linear equation is a *singleton set.*

Equations having the same solution set are called *equivalent equations.*

For example: $4x - 3 = 9$, $4x - 9 = 3$, $4x = 12$ are equivalent equations.

Rules for solving linear equations in one variable

To solve a linear equation in one variable, we transform the given equation into an equivalent equation of the form $x = k$, where $k \in R$. Then the solution set is $S = \{x : x = k\}$ or $\{k\}$.

In an equation, the two sides are equal, *i.e.*, balanced. Performing the same mathematical operations on *both sides* of the equation does not disturb the balance and an equivalent equation is generated.

Example 3. *Solve 6x – 7 = 23 and check your solution.*

Solution. Given: $6x - 7 = 23$

Adding 7 to both sides

$6x - 7 + 7 = 23 + 7$

$\Rightarrow \quad 6x = 30$

OR

Transposing –7 to RHS

$6x = 23 + 7$

$\Rightarrow \quad 6x = 30$

Dividing both sides by 6, we get

$$\frac{6x}{6} = \frac{30}{6} \Rightarrow x = 5 \text{ is the required solution.}$$

The solution set is S = $\{x : x = 5\}$ or $\{5\}$.

Check: Replacing x by 5 in the given equation, we have

$$6 \times 5 - 7 = 23$$

or $$30 - 7 = 23, \quad \text{which is true.}$$

Example 4. *Solve:* $\frac{3x}{5} - \frac{x}{3} = \frac{x}{6} + 1\frac{1}{2}$.

Solution. Given: $\frac{3x}{5} - \frac{x}{3} = \frac{x}{6} + \frac{3}{2}$ $\left[\because 1\frac{1}{2} = \frac{3}{2}\right]$

Multiplying by 30, the LCM of 5, 3, 6 and 2, we get

$$6(3x) - 10x = 5x + 15(3)$$

$\Rightarrow$ $18x - 10x = 5x + 45 \Rightarrow 8x = 5x + 45$

$\Rightarrow$ $8x - 5x = 45 \Rightarrow 3x = 45 \Rightarrow x = \frac{45}{3} = 15.$

The solution set is S = $\{15\}$.

Example 5. *Solve:* $\frac{5x-2}{3} = \frac{3x+2}{2}$.

Solution. Given: $\frac{5x-2}{3} = \frac{3x+2}{2}$.

By cross-multiplication (or multiplying both sides by 6, the LCM of 3 and 2), we have

$$2(5x - 2) = 3(3x + 2)$$

$\Rightarrow$ $10x - 4 = 9x + 6$

$\Rightarrow$ $10x - 9x = 6 + 4$

$\Rightarrow$ $x = 10$, is the required solution.

$\therefore$ The solution set is S = $\{10\}$.

EXERCISE 6.2

1. Verify that $x = 6$ is a solution of the equation

$$2(x - 3) - 17 = 13 - 3(x + 2).$$

2. Verify that $x = 8$ is a solution of the equation $\frac{5x-4}{8} - \frac{x-3}{5} = \frac{x+6}{4}$.

3. Solve the following equations and check your solution:

(*a*) $4x - 7 = 9$ (*b*) $3x + 11 = 2$

(*c*) $7 - 2(5 - 3x) = 4(x - 3) + 5$ (*d*) $7(x - 2) = 2(2x - 4)$

4. Solve the following equations and check your solution:

(*a*) $\frac{5x}{3} + \frac{2}{5} = 1$ (*b*) $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 13$

(*c*) $\frac{2x-1}{3} - \frac{6x-2}{5} = \frac{1}{3}$ (*d*) $\frac{2x-3}{6} - \frac{x-5}{2} = \frac{x}{6}$

6.3 WORD PROBLEMS INVOLVING LINEAR EQUATIONS

Problems stated in words are called *word problems.* Solving word problems involves two steps:

(*i*) formulation (*ii*) solution

The following steps should be followed to solve a word problem.

(*a*) Read the problem carefully and note what is given and what is required.

(*b*) Denote the unknown quantity by x.

(*c*) Translate the statement of the problem into mathematical statements.

(*d*) Use the given conditions to form an equation.

(*e*) Solve the equation.

Example 6. *A number is such that it is as much greater than 84 as it is smaller than 108. Find the number.*

Solution. Let the number be x. Then, the number is greater than 84 by $x - 84$. Also, the number is smaller than 108 by $108 - x$.

Given $x - 84 = 108 - x$

$\Rightarrow$ $x + x = 108 + 84$

$\Rightarrow$ $2x = 192$

$\Rightarrow$ $x = \frac{192}{2} = 96$

Hence the required number is 96.

Example 7. *The present age of Ella's mother is three times the present age of Ella. After 5 years their ages will add to 66 years. Find their present ages.*

Solution. Let Ella's present age be x years, then the present age of Ella's mother is $3x$ years.

After 5 years, Ella's age will be $(x + 5)$ years and her mother's age will be $(3x + 5)$ years.

By the given condition

$$(x + 5) + (3x + 5) = 66 \quad \Rightarrow \quad x + 5 + 3x + 5 = 66$$

$$\Rightarrow \quad 4x + 10 = 66 \quad \Rightarrow \quad 4x = 66 - 10$$

$$\Rightarrow \quad 4x = 56 \quad \Rightarrow \quad x = \frac{56}{4} = 14$$

Therefore, the present age of Ella = 14 years and the present age of Ella's mother = 3 × 14 = 42 years.

Example 8. *The denominator of a fraction exceeds its numerator by 4. If the numerator and the denominator both are increased by 3, the fraction becomes* $\frac{4}{5}$. *Find the original fraction.*

Solution. Let the numerator of the original fraction be x, then its denominator is $(x + 4)$.

The original fraction = $\frac{x}{x+4}$

When the numerator and denominator both are increased by 3, the new fraction = $\frac{x+3}{(x+4)+3} = \frac{x+3}{x+7}$

By the given condition

$$\frac{x+3}{x+7} = \frac{4}{5}$$

By cross-multiplication

$$5(x + 3) = 4(x + 7) \quad \Rightarrow \quad 5x + 15 = 4x + 28$$

$$\Rightarrow \quad 5x - 4x = 28 - 15 \quad \Rightarrow \quad x = 13$$

Therefore, the original fraction = $\frac{13}{13+4} = \frac{13}{17}$.

EXERCISE 6.3

1. If $\frac{1}{3}$ of a number is added to $\frac{1}{5}$ of the same number, the result is 8. Find the number.
2. The sum of three consecutive even numbers is 24. Find the numbers.

3. When a certain number is subtracted from 10 and the result is multiplied by 2, the final result is 4. Find the number.

4. The sum of five consecutive odd numbers is 35. Find the numbers.

5. $\frac{5}{6}$ of the number of pupils in a class is 4 greater than three-quarters of the number in the class. Find the number of pupils in the class.

6. A certain car covers 10 km at a certain speed. If this average speed is reduced by 30 km/h, the car takes the same time to cover a distance of 6 km. Find the speed of the car in the first part of the journey.

6.4 LINEAR INEQUALITIES IN ONE VARIABLE

Two algebraic expressions related by the symbol < (is less than) or > (is greater than) or ≤ (is less than or equal to) or ≥ (is greater than or equal to) form an *inequality.*

An inequality in any one of the following forms:

$$\left.\begin{array}{ll} ax + b < 0 & \ldots(1) \\ ax + b > 0 & \ldots(2) \\ ax + b \leq 0 & \ldots(3) \\ ax + b \geq 0 & \ldots(4) \end{array}\right\} \quad \begin{array}{l} a, b \in R \\ a \neq 0 \end{array}$$

is called a *linear inequality in one variable x.*

(1) and (2) are called *strict inequalities* while (3) and (4) are called *slack inequalities.*

Thus, $2x + 3 < 0$, $3x - 4 > 0$, $7x - 2 \leq 0$

$5x + 2 \geq 0$ are linear inequalities in x.

6.5 SOLVING LINEAR INEQUALITIES IN ONE VARIABLE

To solve a linear inequality in one variable, we transform the given inequality into an equivalent inequality of the form $x < k$ or $x > k$ or $x \leq k$ or $x \geq k$ by the following rules:

(*i*) If the same quantity is added to or subtracted from both sides of an inequality, then the sign of the inequality is not affected.

For example:

$$a < b \;\Rightarrow\; a + c < b + c$$
$$a \geq b \;\Rightarrow\; a - c \geq b - c$$

(*ii*) If both sides of an inequality are multiplied or divided by a *positive number,* then the sign of inequality is not affected.

For example:

$$a \geq b \quad \text{and} \quad c > 0 \quad \Rightarrow \quad ac \geq bc$$

$$a < b \quad \text{and} \quad c > 0 \quad \Rightarrow \quad \frac{a}{c} < \frac{b}{c}.$$

(*iii*) If both sides of an inequality are multiplied or divided by a *negative number,* then the direction of inequality is reversed, *i.e.,* < changes into > and vice versa.

For example:

$$a < b \quad \text{and} \quad c < 0 \quad \Rightarrow \quad ac > bc$$

$$a \geq b \quad \text{and} \quad c < 0 \quad \Rightarrow \quad \frac{a}{c} \leq \frac{b}{c}.$$

Thus, always reverse the sign of inequality when multiplying or dividing by a negative number.

(*iv*) A term may be transposed from one side of the inequality to the other side by changing its sign.

Example 9. *Solve for x:* $-5 < 9 - 2x$.

Solution. $-5 < 9 - 2x$

$\therefore$ $-5 + 2x < 9 - 2x + 2x$ [Adding $2x$ to both sides]

$\therefore$ $2x - 5 < 9$

$\therefore$ $2x - 5 + 5 < 9 + 5$ [Adding 5 to both sides]

$\therefore$ $2x < 14$

$\therefore$ $\frac{2x}{2} < \frac{14}{2}$ [Dividing both sides by 2]

$\therefore$ $x < 7$

Check: If $x = 5$ then $-5 < 9 - 2 \times 5$, *i.e.,* $-5 < -1$ which is true.

Example 10. *Solve for x:* $3 - 5x \geq 2x + 7$.

Solution. $3 - 5x \geq 2x + 7$

$\therefore$ $3 - 5x - 2x \geq 2x + 7 - 2x$ [Subtracting $2x$ from both sides]

$\therefore$ $3 - 7x \geq 7$

$\therefore$ $3 - 7x - 3 \geq 7 - 3$ [Subtracting 3 from both sides]

$\therefore$ $-7x \geq 4$

$$\therefore \qquad \frac{-7x}{-7} \le \frac{4}{-7}$$

[Dividing both sides by –7, so reverse the sign]

$$\therefore \qquad x \le \frac{4}{7}$$

Check: If $x = -1$ then $3 - 5 \times (-1) \ge 2 \times (-1) + 7$, *i.e.*, $8 \ge 5$ which is true.

6.6 GRAPH OF LINEAR INEQUALITIES IN ONE GRAPH

Solutions of inequalities can be represented on a number line as follows:

(*i*) Graph of $x < 5$, $x \in$ N

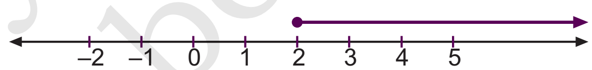
The solution set = {1, 2, 3, 4} is shown by thick dots on the number line. It consists of four isolated points.

(*ii*) Graph of $-3 \le x < 4$, $x \in$ I

The solution set = {–3, –2, –1, 0, 1, 2, 3} is shown by thick dots on the number line.

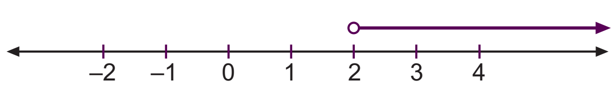
(*iii*) Graph of $x \ge 2$, $x \in$ R

The solution set = $\{x : x \ge 2, x \in \text{R}\}$ is shown by a rightward arrow emanating from 2. The shaded circle above 2 indicates '2 is included'.



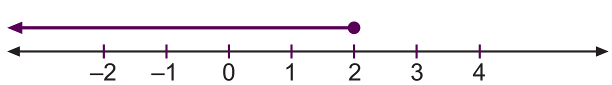
(*iv*) Graph of $x > 2$, $x \in$ R

The solution set = $\{x : x > 2, x \in \text{R}\}$ is shown by a rightward arrow emanating from 2. The unshaded circle above 2 indicates '2 is not included'.



(*v*) Graph of $x \le 2$, $x \in$ R

The solution set = $\{x : x \le 2, x \in \text{R}\}$ is shown by a leftward arrow emanating from 2. The shaded circle above 2 indicates '2 is included'.



(*vi*) Graph of $x < 2$, $x \in$ R

The solution set = $\{x : x < 2, x \in R\}$ is shown by a leftward arrow emanating from 2. The unshaded circle above 2 indicates '2 is not included'.

(*vii*) Graph of $-2 < x \leq 3$, $x \in$ R

The solution set = $\{x : -2 < x \leq 3, x \in R\}$ is shown by a line segment whose left end point above –2 is not included (shown by an unshaded circle) and right end point above 3 is included (shown by a shaded circle).

Example 11. *Solve: $3x + 1 \leq 16$, $x \in$ **N** and represent the solution on the number line.*

Solution. Given $\quad 3x + 1 \leq 16$

Transposing 1 to RHS

$$3x \leq 16 - 1 \quad \Rightarrow \quad 3x \leq 15$$

Dividing both sides by 3,

$$x \leq 5$$

Since, $\quad x \in$ **N**, therefore, $x = 1, 2, 3, 4, 5$

The solution set = {1, 2, 3, 4, 5} is shown by thick dots on the number line.

Example 12. *Solve: $3(x - 1) > 2(x + 2) - 9$ and represent the solution on the number line.*

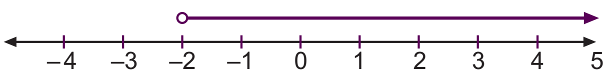
Solution. Given $\quad 3(x - 1) > 2(x + 2) - 9$

$$\Rightarrow \quad 3x - 3 > 2x + 4 - 9 \quad \Rightarrow \quad 3x - 3 > 2x - 5$$

Collecting the terms of x on LHS and the constant terms on RHS

$$3x - 2x > -5 + 3 \quad \Rightarrow \quad x > -2$$

The solution set = $\{x : x > -2, x \in R\}$ is shown by a rightward arrow emanating from –2. The unshaded circle above –2 indicates '–2 is not included'.



Example 13. *Solve:* $\frac{2x-1}{3} \le \frac{3x-2}{4} - \frac{2-x}{5}$.

Represent the solution on the number line.

Solution. Given $\frac{2x-1}{3} \le \frac{3x-2}{4} - \frac{2-x}{5}$.

Multiplying both sides by 60, the LCM of 3, 4, 5

$$20(2x-1) \le 15(3x-2) - 12(2-x)$$

$$\Rightarrow \quad 40x - 20 \le 45x - 30 - 24 + 12x$$

$$\Rightarrow \quad 40x - 20 \le 57x - 54$$

Collecting the terms of x on LHS and the constants on RHS

$$40x - 57x \le -54 + 20$$

$$\Rightarrow \quad -17x \le -34$$

Dividing both sides by -17, which is negative

$$x \ge 2$$

The solution set = $\{x : x \ge 2, x \in R\}$ is shown by a rightward arrow emanating from 2. The shaded circle above 2 indicates '2 is included'.

Example 14. *Solve: $0 \le 3x - 1 \le 2$. Represent the solution on the number line.*

Solution. Given $0 \le 3x - 1 \le 2$

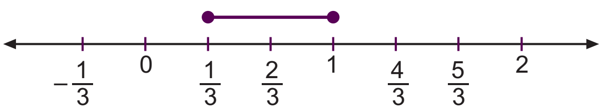
This is a double inequality.

Adding 1 throughout, $1 \le 3x \le 3$

Dividing by 3 throughout, $\frac{1}{3} \le x \le 1$

The solution set = $\left\{x : \frac{1}{3} \le x \le 1, x \in R\right\}$ is shown by a line segment.

The circles above $\frac{1}{3}$ and 1 are both shaded because $\frac{1}{3}$ and 1 are both included in the solution set.



EXERCISE 6.4

1. Solve the following inequalities and represent the solution on the number line:

(*a*) $5x - 3 < 7,\ x \in \mathbf{N}$ (*b*) $3 - 2x \geq x - 10,\ x \in \mathbf{N}$

(*c*) $11 + 2x > 5,\ x \in \mathbf{I}$ (*d*) $4x + 3 < 6x + 7,\ x \in \mathbf{I}$

2. Solve the following inequalities and represent the solution on the number line:

(*a*) $\frac{5x}{2} + \frac{3x}{4} \geq \frac{39}{4}$ (*b*) $\frac{x}{4} - \frac{3}{5} \leq \frac{x}{2} + 1$

(*c*) $\frac{2x-1}{3} \geq \frac{3x-2}{4} - \frac{2-x}{5}$ (*d*) $1 \leq 3(x - 2) + 4 < 10,\ x \in \mathbf{N}$

3. Illustrate the solution set of the linear inequality $6 - 3x > x + 12$ when x is an integer.

4. Illustrate the solution set of the linear inequality $x + 1 < 4$ on the number line.

5. Find the solution set for the linear inequality $3x + 8 > 2$ and illustrate it on the number line when

(*a*) x is an integer (*b*) x is a real number

6. Find the solution set for linear inequality $7x + 3 > 5x + 9$ and illustrate your answer on the number line, where x is a real number.

7. Illustrate the solution set of the linear inequality $-4 < x \leq 5$ on the "number line.

6.7 WORD PROBLEMS INVOLVING LINEAR INEQUALITIES

Solving word problems involving linear inequalities in one variable involves two steps:

(*i*) formulation (*ii*) solution

Read the problem carefully. Denote the unknown quantity by x. Translate the statement of the problem into mathematical statements. Use the given conditions to form an inequality and then solve it.

Example 15. *Find all pairs of consecutive odd natural numbers which are greater than 12 and their sum is less than 50.*

Solution. Let x and $x + 2$ be two consecutive odd natural numbers.

Given: $x > 12$ and $x + (x + 2) < 50$

$\Rightarrow$ $x > 12$ and $x + x + 2 < 50$

$\Rightarrow \quad x > 12 \quad \text{and} \quad 2x < 50 - 2$

$\Rightarrow \quad x > 12 \quad \text{and} \quad 2x < 48$

$\Rightarrow \quad x > 12 \quad \text{and} \quad x < 24$

$\Rightarrow \quad 12 < x < 24$

Since x is an odd natural number, therefore,

$$x = 13, 15, 17, 19, 21, 23$$

Hence, the required possible pairs are $(x, x + 2)$

$$= (13, 15), (15, 17), (17, 19), (19, 21), (21, 23), (23, 25).$$

Example 16. *On a day, temperature in a city changes from 30° to 35° Celsius (C) in one hour. Find the range of change of temperature in degree Fahrenheit (F) if conversion formula is given by* $C = \frac{5}{9}(F - 32)$.

Solution. Given $30 \le C \le 35$

Putting $C = \frac{5}{9}(F - 32)$, we have

$$30 \le \frac{5}{9}(F - 32) \le 35$$

Multiplying throughout by 9,

$$270 \le 5(F - 32) \le 315$$

Dividing throughout by 5,

$$54 \le F - 32 \le 63$$

Adding 32 throughout

$$86 \le F \le 95$$

Therefore, the temperature changes from 86°F to 95°F.

EXERCISE 6.5

1. One-fourth of a number added to one-fifth of the same number is less than or equal to 18. Find the range of values of the number.
2. A petty trader wants to buy pineapples at ¢25 each and apples at ¢20 each. She decides to buy twice as many apples as pineapples. Her total cost was not less than L$ 6.50 and not more than L$ 7.80.

   Taking x to be the number of pineapples:

   (*a*) Write the given information as an inequality in x.

   (*b*) Find the truth set of the inequality,

   (*c*) Illustrate the solution set on the number line.

3. Henry scored 78 on his mid-semester examination in Mathematics. If he is to get a grade A, the average of his mid-semester and final examination must be between 80 and 88 inclusive. In what range must his final examination score lie to get a grade A?

REVIEW EXERCISE

1. Solve for x: $11 - 5x = 26$.
2. Solve for x: $\frac{x}{3} + 2 = -2$.
3. Solve for x:
   (*a*) $4 = 3 - 2x$ (*b*) $13 = -1 - 7x$
4. Solve for x:
   (*a*) $4 = \frac{2+x}{3}$ (*b*) $-1 + \frac{x}{3} = 7$
5. Solve the following equations and check your solution:
   (*a*) $5x - 3 = 3x + 5$ (*b*) $3(x - 1) = x - 11$
   (*c*) $3(2x - 1) = 5 - (3x - 2)$ (*d*) $2(x - 1) + 3 = x - 3(x + 1)$
6. Solve the following equations and check your solution:
   (*a*) $\frac{2x}{3} - \frac{3x}{8} = \frac{7}{12}$ (*b*) $\frac{x-1}{3} - \frac{x-2}{4} = 1$
7. Shelia is four times as old as Albert. In ten years time, Shelia will be twice as old as Albert. Find their ages.
8. A boy has to cover 4 km to catch a bus. He walks part of the distance at 3 km/h and runs the rest at 5 km/h. If he takes 1 hour to complete the distance, for how many kilometres does he walk?
9. Solve the following inequalities and represent the solution on the number line:
   (*a*) $8x + 3 > 3(2x + 1) + x + 5$ (*b*) $3(x - 2) \geq 2x - 3$
10. Solve the following inequalities and represent the solution on the number line:
    (*a*) $\frac{3x-4}{2} \geq \frac{x+1}{4} - 1$ (*b*) $\frac{x}{4} < \frac{5x-2}{3} - \frac{7x-3}{5}$
11. 4 is added to three times a certain number, the result is not more than when twice that number added to 10. Find the possible range of values of the number.
12. The result of doubling a number is at most 20. What is the number?

MULTIPLE CHOICE QUESTIONS (MCQs)

1. The solution of the linear equation $-2(x + 3) = 8$ is

(a) $\frac{3}{2}$ (b) $\frac{2}{3}$ (c) $\frac{9}{2}$ (d) None of these

2. The linear inequality $3x - 4 > 17$ has a solution given by

(a) $x > 7$ (b) $x \geq 7$ (c) $x < 7$ (d) $x \leq 7$

3. The value of x which satisfies the linear inequality $4x + 3 < 6x + 7$ is

(a) $x < -2$ (b) $x > -2$ (c) $x < 2$ (d) $x > 2$

4. The solution set of the linear inequality $5x - 3 < 7$ where x is an integer is

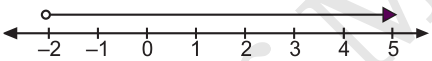
(a) $\{-4, -3, -2, -1, 0, 1\}$ (b) $\{\ldots, -4, -3, -2, -1, 0, 1\}$

(c) both (a) and (b) (d) None of these

5. The number line representing the solution set for the linear inequality $3x + 8 > 2$; x is a real number is

(a) 

(b) 

(c) 

(d) None of these

6. Albert says that he has 7 marbles more than five times the marbles Ella has. Albert has 37 marbles. How many marbles does Ella has?

(a) 7 (b) 8 (c) 6 (d) None of these

7. Islah's father is 49 years old. He is 4 years older than three times Islah age. The age of Islah's is

(a) 15 years (b) 19 years (c) 11 years (d) None of these

8. A teacher tells his class that the highest marks obtained by a pupil in his class is twice the lowest marks plus 7. The highest score is 87. The lowest score in the class is

(a) 80 (b) 47 (c) 87 (d) 40

9. Daniel subtracts thrice the number of notebooks he has from 50, he finds the result to be 8. The number of notebooks Daniel has

(a) 24 (b) 42 (c) 14 (d) None of these

10. The solution set of the linear inequality $5x + 8 < 23$; x is a whole number is

(a) $\{1, 2\}$ (b) $\{0, 1\}$ (c) $\{0, 2\}$ (d) $\{0, 1, 2\}$

RECAP AT A GLANCE

- Two equations that have the same solution are called *equivalent* equations.
- Perform operations on both sides of each equation in order to *isolate the unknown.*
- *Solving an equation* is the process of finding the solutions of the equation.
- The set of all solutions of an equation is called the *solution set* of the equation.
- Two algebraic expressions related by the symbol < (is less than) or > (is greater than) or ≤ (is less than or equal to) or ≥ (is greater than or equal to) form an *inequality.*
- Problems stated in words are called *word problems.*
- Equations having the same solution set are called *equivalent equations.*
- If the same quantity is added to or subtracted from both sides of an inequality, then the sign of the inequality is not affected.
- If both sides of an inequality are multiplied or divided by a *positive number,* then the sign of inequality is not affected.
- If both sides of an inequality are multiplied or divided by a *negative number,* then the direction of inequality is reversed.
- A term may be transposed from one side of the inequality to the other side by changing its sign.
- Solutions of inequalities can be represented on a number line

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PERIOD-IV

M10CH7

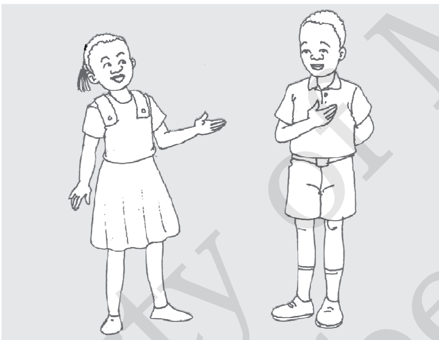
TOPIC 7

Relations and Functions

7.1 RELATIONS

In our daily life, we come across many patterns that characterize relations such as brother and sister, father and son, teacher and pupil, etc. A ‘relation is just a relationship between two sets of information’.

For example:



Ella “is the sister of” Felix.

Jerelyn “is the teacher of” Samuel.

Here, the pairing of the names of first person and second person is a relation. In these relations, the pairs of names of first person and second person are ‘*ordered*’, which means one comes first and the other comes second and this order of each pair cannot be changed.

Realizing the Relations in Mathematics

In mathematics also, we come across many relations such as the *numeral* 7 “is less than” the *numeral* 11, line l “is parallel to” line m, set A “is a subset of” set B.

Consider the following examples of relation:

2 “is half of” 4. 3 “is the square root of” 9.

5 “is less than” 8.

$\frac{8}{10}$ "is an equivalent fraction of" $\frac{4}{5}$.

In all these examples, we notice that a relation involves pairs of objects in certain order.

Example 1. *Find the relation between the following sets by considering the numbers opposite to each other:*

Solution. By considering the numbers opposite to each other, we find that

1 is one third of 3, 2 is one third of 6,

3 is one third of 9, 4 is one third of 12,

5 is one third of 15.

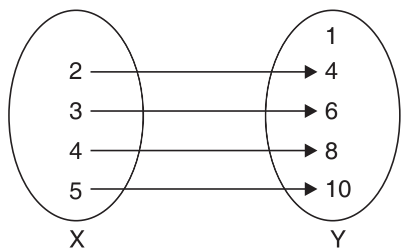
Thus in general, every element of set A is one third of its opposite element in set B.

The relation is therefore "is one third of".

Representing a Relation as a Mapping

A *relation* can be represented by matching diagram.

Consider the following matching diagram:



What do you observe?

2 is half of 4, 3 is half of 6,

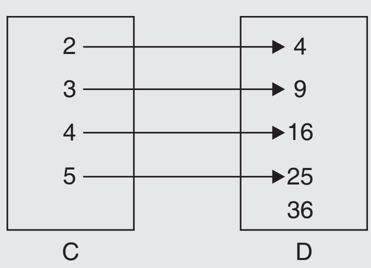
4 is half of 8, 5 is half of 10.

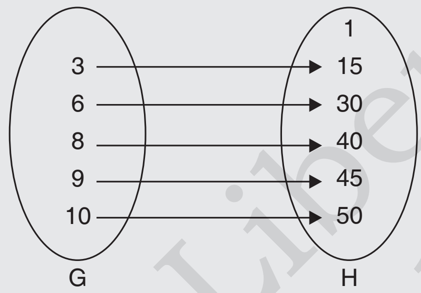
Thus in general, every element of set X is half of its corresponding element in set B.

The relation is therefore "is half of".

ACTIVITY 1

Consider the following relations:





What are the two sets shown?

What is the relation between the two sets C and D?

What about elements in set G and set H?

We observe that in the sets C and D:

2 is square root of 4, 3 is square root of 9,

4 is square root of 16, 5 is square root of 25.

Thus, C and D have the relation "*is a square root of*". From sets G and H, we observe that,

3 is one fifth of 15, 6 is one fifth of 30,

8 is one fifth of 40, 9 is one fifth of 45,

10 is one fifth of 50.

Thus, G and H have the relation "*is one fifth of*".

From the above discussion, we find that all the objects from the first set are matched with *unique* objects in the second set (*i.e.*, there cannot exist two or more objects in the second set for any object in the first set). There may be some objects in the second set which have no link with any object of the first set.

Note: **1.** A relation can be represented by matching diagram.

2. A mapping is a visual representation of a relation.

Domain, Co-Domain and Range of a Relation

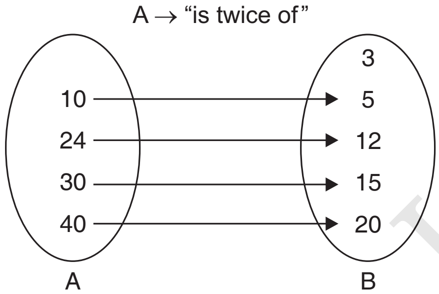
Domain

Domain of a relation is the set of all elements in the *first set* from the direction of the arrow diagram. It is represented by **D**.

Co-domain

Co-domain of a relation is the set of elements in the *second set* (*i.e.*, the whole second set) from the direction of the arrow diagram.

For example: Consider the following relation:



We observe that:

The set of elements in domain D = {10, 24, 30, 40}

The set of elements in co-domain is {3, 5, 12, 15, 20}

Range

In every relation, an element *a* in set A is related or (mapped onto) an element *b* in set B. Set A is domain and set B the co-domain to this type of relation, the element *b* in the co-domain is called the image of *a* in the domain A. The set of all the images of the domain is referred to as the range.

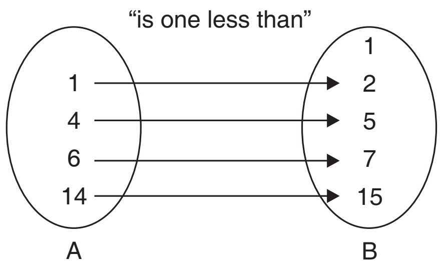
The range is therefore a subset of the co-domain and denoted by R.

For example, from the relation "*is twice of*" shown in the above diagram:

The set of elements in range R = {5, 12, 15, 20}

Example 2. *The following arrow diagram shows a relation between the two sets.*

Find the domain, co-domain and range.



Solution. The domain D = {1, 4, 6, 14}

The co-domain is {1, 2, 5, 7, 15}

The range R = {2, 5, 7, 15}.

Note:
1. Domain is the subset of elements in the first set from the direction of the matching diagram.
2. Co-domain is the set of elements in the second set (*i.e.*, the whole second set) from the direction of the matching diagram.
3. Range is a subset of the co-domain.

Relation as Ordered Pair

An *ordered pair* is a pair of objects taken in a specific order. An ordered pair is written by listing its two members in a specific order, separating them by a comma and enclosing the pair in parentheses. In the ordered pair (a, b), a is called the *first member* (or *component*) and b the *second member* (or *component*).

ACTIVITY 2

Suppose Daniel has two children—Felix and Ella. The ordered pairs in which the first component is father and the second component is a child are (Daniel, Felix) and (Daniel, Ella).

How will you interpret (Felix, Daniel) or (Ella, Daniel)?

Does the interpretation make sense?

Is (Daniel, Felix) = (Daniel, Ella)?

No, because the second member, *i.e.*, Felix and Ella are different.

Two ordered pairs are equal only when the corresponding components are equal.

Thus, $(a, b) = (c, d)$ only when $a = c$ and $b = d$.

Clearly, $(a, b) = (b, a)$ only when $a = b$.

Note: **1.** The two components of an ordered pair may be equal.

2. Remember that $\{a, b\} \neq (a, b)$, because $\{a, b\}$ is a set whereas (a, b) is an ordered pair.

Example 3. *If $(a, 2) = (3, 2)$, find the value of a.*

Solution. Since the ordered pairs are equal, the corresponding elements are equal.

Therefore, $a = 3$.

Example 4. *If $(x - 3, 5) = (2, y + 1)$, then find x and y.*

Solution. Here, $(x - 3, 5) = (2, y + 1)$

Equating corresponding components,

$$x - 3 = 2 \quad \text{and} \quad 5 = y + 1$$

$$\Rightarrow \quad x = 5 \quad \text{and} \quad y = 4$$

EXERCISE 7.1

1. Identify and write the relation between the following pairs of sets:

(*a*) Monrovia .... Liberia

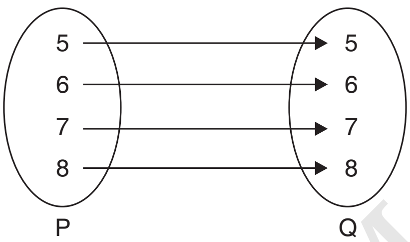
(*b*) Mathematics .... High School

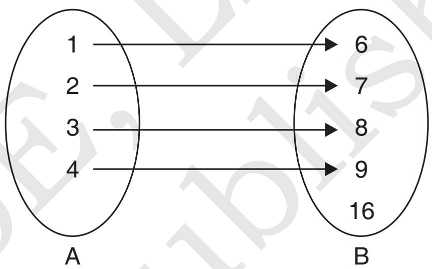
(*c*) English .... Liberia

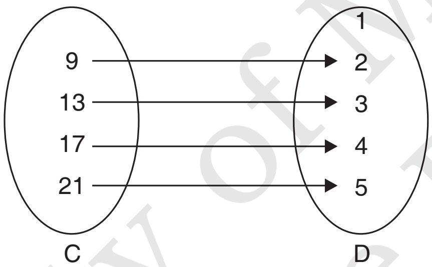
(*d*) Nelson Mandela .... South Africa

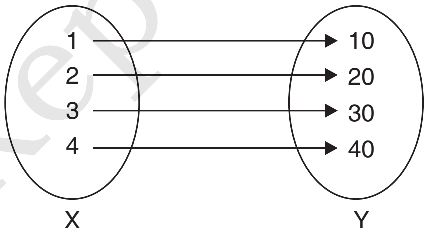
(*e*) L$ .... Liberia

2. Identify the following relations between the given pairs of sets:

(*a*)


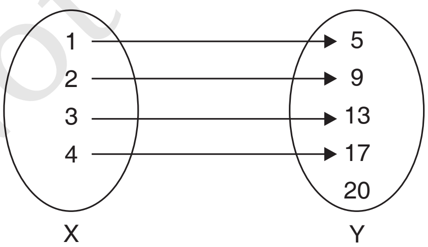
(*b*)


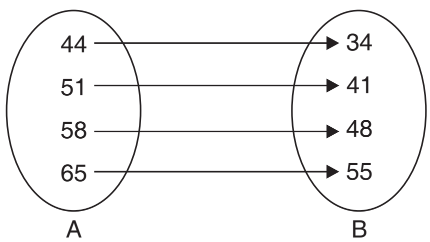
(*c*)


(*d*)


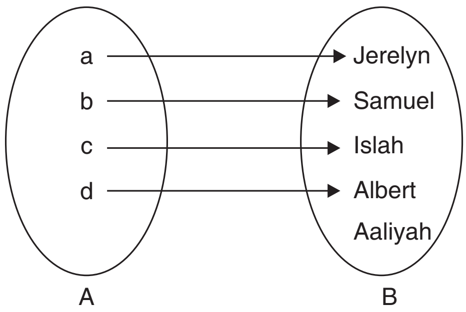
3. Given P = {Islah, Felix, Ella, Thomas},
Q = {Days of the week}
The relation is "was born on".
Draw an arrow diagram between the pairs of sets (assume a day of your choice for the birth of each).

4. Find the domain, co-domain and range of the following relations:

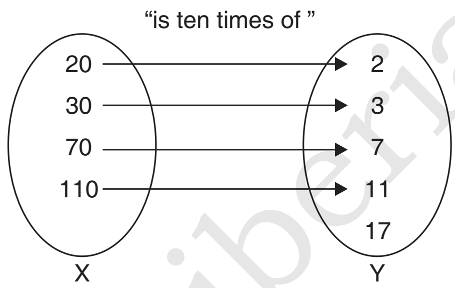
(*a*)


(*b*)


5. Write a set of ordered pairs of members that satisfy the following relation:



Q. 5



Q. 6

6. Write a set of ordered pairs of members that satisfy the above relation.

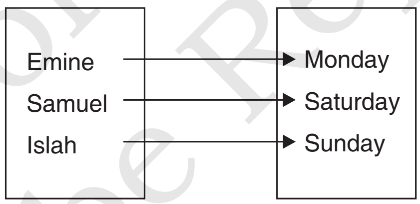
7. If $(x + 2, y - 4) = (5, 2)$, find the values of x and y.

7.2 TYPES OF RELATIONS

One-to-One Relation

In this type of relation, the first set (domain) is mapped to only one element on the second set (co-domain).

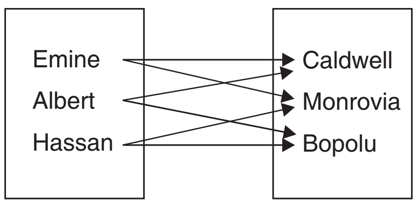
For example:



One-to-Many Relation

Here, each element in the domain is mapped to more than one element in the co-domain.

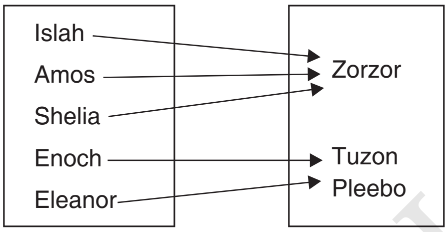
For example:



Many-to-One Relation

Here many elements in the domain are mapped to one element in the co-domain.

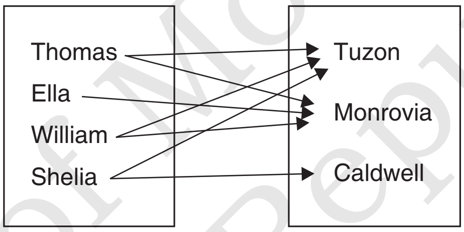
For example:



Many-to-Many Relation

In this relation, several elements in the domain are mapped to more than one element in the co-domain.

For example:



Example 5. *Let A = {1, 2, 3, 4, 5} and B = {2, 4, 6, 8}. Which of the following relations from A to B is one-to-one, one-to-many, many-to-one, many-to-many? Draw an arrow diagram in each case.*

(*i*) $R_1 = \{(3, 4), (4, 4), (5, 4), (2, 6)\}$ (*ii*) $R_2 = \{(2, 4), (3, 6), (4, 8)\}$

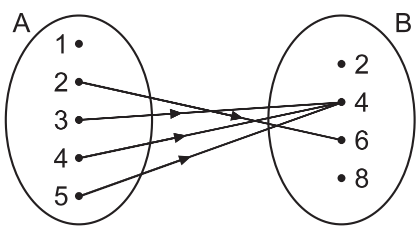
(*iii*) $R_3 = \{(2, 4), (2, 6), (2, 8), (5, 6)\}$ (*iv*) $R_4 = \{(2, 4), (3, 6), (3, 8), (4, 6), (4, 8), (5, 8)\}$

Solution.

(*i*) $R_1 = \{(3, 4), (4, 4), (5, 4), (2, 6)\}$

Here *many* elements 3, 4, 5 in the domain of R_1 are associated with one element 4 of B.

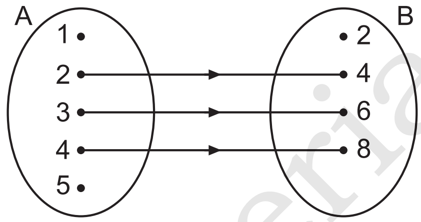
$\Rightarrow$ R_1 is a many-to-one relation.



(*ii*) $R_2 = \{(2, 4), (3, 6), (4, 8)\}$

Here every element in the domain of R_2 is associated with a unique element of B.

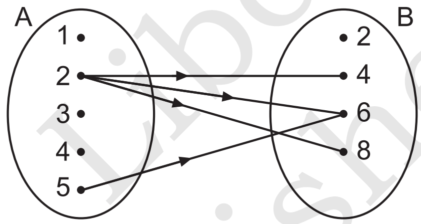
$\Rightarrow$ R_2 is a one-to-one relation.



(*iii*) $R_3 = \{(2, 4), (2, 6), (2, 8), (3, 6)\}$

Here an element 2 in the domain of R_3 is associated with many elements 4, 6, 8 of B.

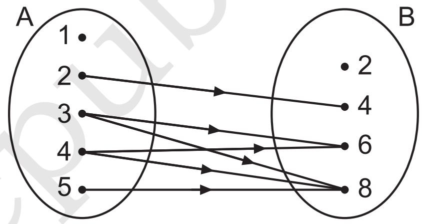
$\Rightarrow$ R_3 is a one-to-many relation.



(*iv*) $R_4 = \{(2, 4), (3, 6), (3, 8), (4, 6), (4, 8), (5, 8)\}$

Here many elements 3, 4 in the domain of R_4 are related to 6 and 8 respectively in B and 6 and 8 have many pre-images in the domain of R_4.

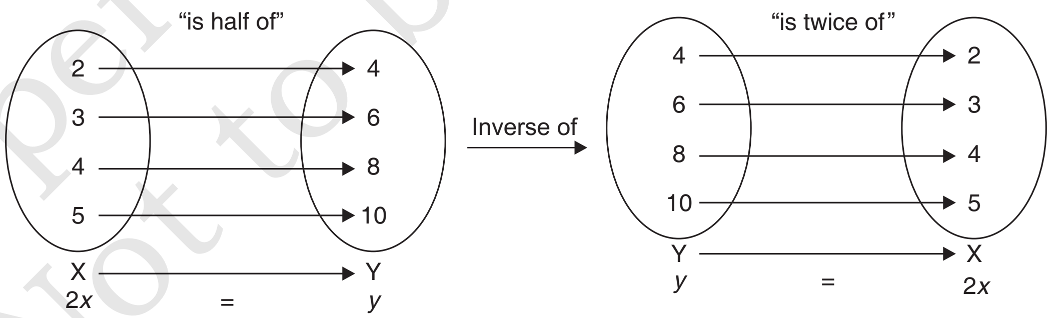
$\Rightarrow$ R_4 is a many-to-many relation.

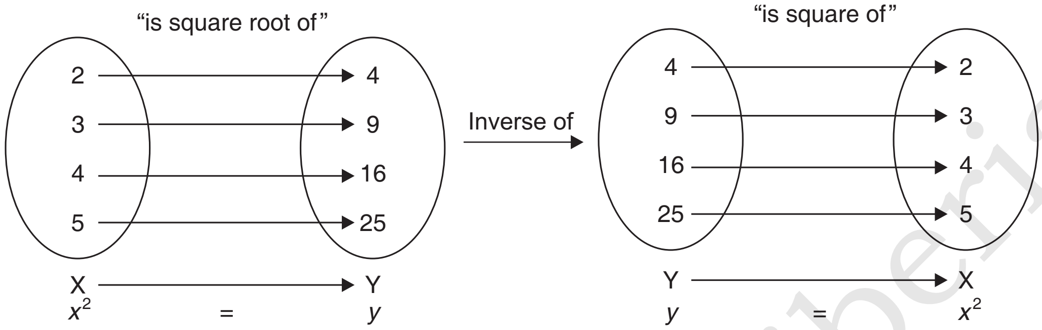


Rules for Mapping

We can state rules for mapping by using the inverse of the relation.

For example, consider the following relation:





Note: The inverse of a relation is possible only when the range equals the co-domain.

If $(x, y) \in R$, we say that x is related to y.

To write the rule, we introduce a variable ordered pair (x, y) and for the rule, y is expressed in terms of x, (*i.e.*, the inverse relation).

For example, the rule of the mapping

$$\{(2, 4), (3, 6), (4, 8), (5, 10)\}$$

is the inverse mapping, which is "is twice" or y is two times x, (*i.e.*, $y = 2x$).

This may be illustrated in the following table:

Domain	2	3	4	5	x
	↓	↓	↓	↓	↓
Range	4	6	8	10	$y = 2x$

Example 6. *Find the rule for the mapping:*

$$\{(2, 1), (3, 4), (4, 7), (5, 10)\}.$$

Solution. The mapping is $\{(2, 1), (3, 4), (4, 7), (5, 10)\}$.

The rule of the mapping is the inverse mapping, which is "five less than thrice" or y is three times x minus five, (*i.e.*, $y = 3x - 5$). This may be illustrated in a table as shown below:

Domain	2	3	4	5	x
	↓	↓	↓	↓	↓
Range	1	4	7	10	$y = 3x - 5$

Finding the value(s) of *x* which make(s) relations undefined

Any expression of the form $\frac{f(x)}{g(x)}$ is *undefined* or *not defined*, if $g(x) = 0$.

Example 7. *Find the value of x which makes the mapping* $x \to \frac{9}{x+5}$ *undefined.*

Solution. Note that any fraction is undefined if the denominator is zero.

Therefore the mapping is undefined if

$$x + 5 = 0 \quad \Rightarrow \quad x = -5$$

EXERCISE 7.2

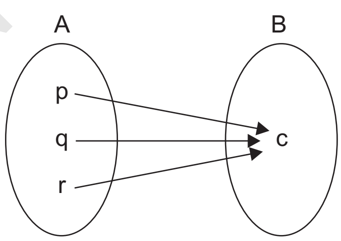
1. Determine the type of the relation in the following:
   (*a*) $R_1 = \{(1, 1), (2, 4), (3, 9), (4, 16)\}$
   (*b*) $R_2 = \{(1, 5), (2, 5), (3, 5), (4, 6)\}$
   (*c*) $R_3 = \{(3, 1), (3, 2), (3, 3), (1, 4), (2, 7)\}$
   (*d*) $R_4 = \{(1, 2), (3, 5), (3, 7), (4, 5), (4, 7), (5, 8)\}$
2. Find the rule for the following mappings:
   (*a*) $\{(2, 12), (3, 18), (4, 24), (5, 30)\}$ (*b*) $\{(1, 5), (2, 8), (3, 11), (4, 14)\}$
   (*c*) $\{(2, 18), (3, 25), (4, 32), (5, 39)\}$ (*d*) $\{(2, 16), (3, 81), (4, 256), (5, 625)\}$
   Show these in the form of tables also.
3. Find the value of *x* which makes the following relations not defined
   (*a*) $x \to \frac{1}{2x^2 - 5x + 2}$ (*b*) $x \to \frac{2x}{x^2 - 5x}$ (*c*) $x \to \frac{1}{4x^2 - 4x + 1}$

7.3 FUNCTIONS

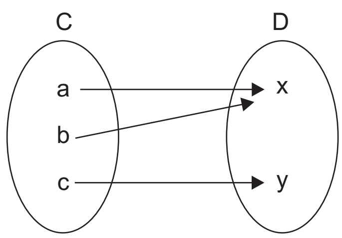
A *function* is a mapping in which each element in the domain is mapped onto one and only one member in the co-domain. Thus, for every element of the domain, there is exactly one image in the co-domain.

For example: (*a*) The following mappings are *functions*.

(*i*)

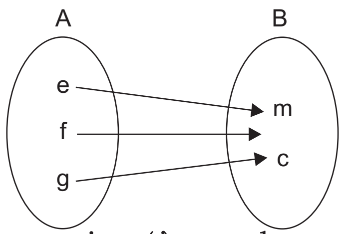


(*ii*)

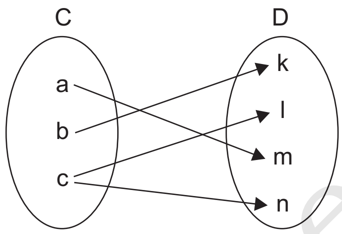


(*b*) The following mappings are *not functions*.

(*i*)



(*ii*)



In mapping (*i*), an element of the domain A (*i.e. f*) has no image.

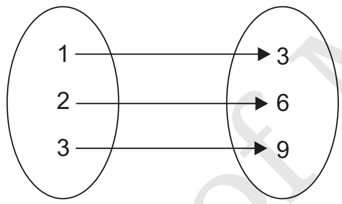
In mapping (*ii*), an element of the domain C (*i.e. c*) has more than one image (*i.e. l* and *n*).

One-to-one function

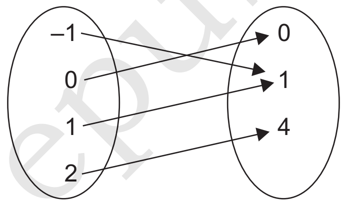
A one-to-one function is a function in which each element in the domain has only *one* image in the co-domain and each element in the co-domain is associated with only one element in the domain.

For example:

(*i*)



(*ii*)



In figure (*i*) above is a one-to-one function.

In figure (*ii*) above is *not* a one-to-one function since two elements of the domain {*i.e.* – 1 and 1} have the same image (*i.e.*, 1).

The Function Notation

If $f(x) = 2x + 3$, then by definition

$f(1) = 2(1) + 3 = 2 + 3 = 5$ (*i.e.* replace x by 1)

$f(2) = 2(2) + 3 = 4 + 3 = 7$ (*i.e.* replace x by 2)

$f(-3) = 2(-3) + 3 = -6 + 3 = -3$ (*i.e.* replace x by – 3)

Note that $f(1)$ means the image of 1 under the function f.

Example 8. *Let f be a function from A to B such that*

$$A = \{1, 2, 3, 4\} \quad \text{and} \quad B = \{3, 5, 7, 9, 11, 13\}$$

If $f(x) = 2x + 1$, find the range of $f(x)$.

Solution. Here $x \in$ A and $f(x) \in$ B.

Given $f(x) = 2x + 1$

i.e., f image of x is $2 \times x + 1$.

Therefore, $f(1) = 2 \times 1 + 1 = 3$ $f(2) = 2 \times 2 + 1 = 5$

$f(3) = 2 \times 3 + 1 = 7$ $f(4) = 2 \times 4 + 1 = 9$

Range of f = set of f images of all elements of A.

$= f(A) = \{f(1), f(2), f(3), f(4)\} = \{3, 5, 7, 9\}$,

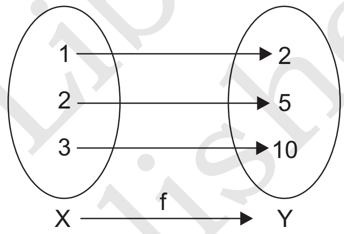
which is always a subset of co-domain B.

Example 9. *A function $f: x \to x^2 + 1$ is defined on the domain {1, 2, 3}. Use arrow diagram to show whether or not f is one-to-one.*

Solution. $f(x) = x^2 + 1$

$\Rightarrow$ $f(1) = 1^2 + 1 = 2$; $f(2) = 2^2 + 1 = 5$;

$f(3) = 3^2 + 1 = 10$



Since each member of the domain {1, 2, 3} has distinct image in the co-domain, the function f is one-to-one.

EXERCISE 7.3

1. A function $g : x \to x^2 + 1$ is defined on the domain {–1, 0, 1, 2}. Use arrow diagram to show whether or not g is one-to-one.
2. A function is defined by $f : x \to \dfrac{2+x}{1+x}$.
   (*a*) Find the image of 1 and 2 under f.
   (*b*) What value of x makes the function undefined?
3. A function is defined by $f: x \to 2x - 3$ on the domain {– 2, – 1, 0, 1, 2}, find the range of the function.
4. The image of x of a function defined by $f : x \to 2x + 5$ is 15, find x.
5. A function $f : x \to ax + b$ is such that $f(1) = 9$ and $f(2) = 14$. Find;
   (*a*) the values of a and b (*b*) $f(-1)$
   (*c*) $f(0)$ (*d*) $f(-2)$

7.4 CHANGE OF SUBJECT

Subject of a formula is the variable which is expressed in terms of other variables involved in the formula.

When a formula involves more than one varables, we can express each variable in terms of the others, *i.e.*, we can make any variable the subject of the formula.

For example: Consider the formula $v = xyz$

Here v is expressed in terms of x, y, z so that v is the subject of this formula.

Writing the formula as $x = \frac{v}{yz}$, x is the subject

Writing the formula as $y = \frac{v}{xz}$, y is the subject

Writing the formula as $z = \frac{v}{xy}$, z is the subject

Thus, in a given formula, we can change the given subject to some other subject of our choice.

Example 10. *Make r the subject of the formula* $V = \frac{1}{3}\pi r^2 h$.

Solution. Given $V = \frac{1}{3}\pi r^2 h$

$\Rightarrow$ $3V = \pi r^2 h$ $\Rightarrow$ $\frac{3V}{\pi h} = r^2$

$\Rightarrow$ $r^2 = \frac{3V}{\pi h}$ $\Rightarrow$ $r = \sqrt{\frac{3V}{\pi h}}$

which is the required formula with subject r.

EXERCISE 7.4

Change the subject of each of the following formulas to the letter given against them:

1. $I = \frac{PRT}{100}$, P

2. $C = 2\pi r$, r

3. $A = \pi r^2$, r

4. $V = 4\pi r^2$, r

5. $v^2 = u^2 + 2as$, s

6. $C = \frac{5}{9}(F - 32)$, F

7.5 GRAPHS OF LINEAR FUNCTIONS

Any function of the form $y = mx + c$, where m and c are constants is called a *linear function* or *linear relation.*

For example: $y = 3x + 1$, $5y = 7x - 2$, and $y = x$ are linear functions.

The graphs of any linear function is a *straight line*. To draw the graph of the linear function $y = 2x + 1$ in the interval $-2 \le x \le 4$, form a table of corresponding values of x and y. Put all the values of x from $x = -2$ to $x = 4$ (*i.e.* $x = -2, -1, 0, 1, 2, 3, 4$) into the given relation and find the corresponding y values i.e. when $x = -2$, $y = 2(-2) + 1 = -4 + 1 = -3$. When $x = -1$, $y = 2(-1) + 1 = -2 + 1 = -1$.

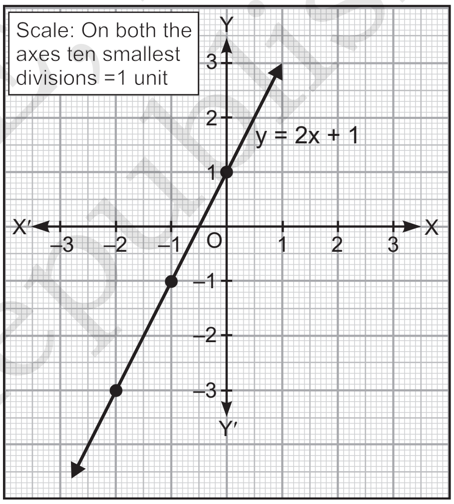
When $x = 0$, $y = 2(0) + 1 = 0 + 1 = 1$ etc. The table below shows the values of x and their corresponding values of y.

x	–2	–1	0	1	2	3	4
y	–3	–1	1	3	5	7	9

Note that you may use only three values of x in the interval given (*i.e.* $x = -2, 0, 4$) since the graph is a straight line.

Note: Here, we need only two points to draw a straight line. But more than two points enables you to check your calculations.

Now, plot the points (–2, –3), (–1, –1), (0, 1) etc. on the graph and draw a straight line through them as shown in the figure.



Example 11. *Using a scale of 2 cm to 1 unit on the x-axis and 2 cm to 2 units on the y-axis, draw the graphs for the straight lines $x + y = 4$ and $3x - y = 8$ on the same graph sheet. From your graphs find the coordinates of the point of intersection.*

Solution. Table for $x + y = 4$

$\Rightarrow \quad y = 4 - x$

x	– 1	1	2
y	5	3	2

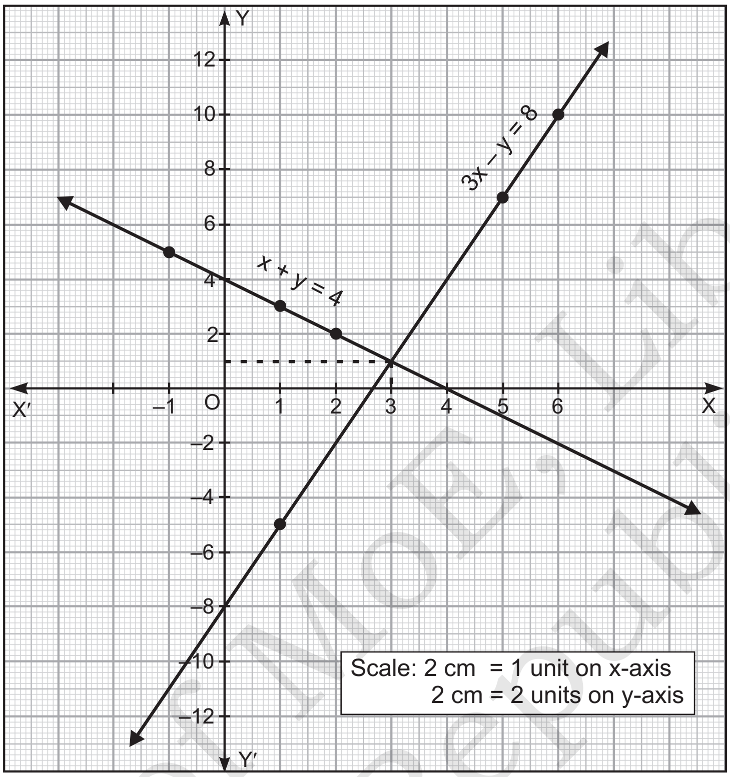
Table for $3x - y = 8$

$\Rightarrow \quad y = 3x - 8$

x	1	5	6
y	–5	7	10

Now, plot the above points on the graph and draw straight lines through them as shown in the figure.

From the graph, the point of intersection is (3, 1).



EXERCISE 7.5

1. Form a table of corresponding values of x and y given $y = 3x + 2$ in the interval $-2 \le x \le 4$. Plot the points on a graph sheet. Using a ruler draw a straight line through the plotted points.

2. Draw the graph of

(*a*) $y = 2x$ (*b*) $y = 2x - 2$ (*c*) $2y = x + 4$ (*d*) $x + 3y + 1 = 0$

(*e*) $x - y + 1 = 0$

3. Draw the graph of the following pair of linear functions.

(*a*) $y = x$ and $y = -x$ (*b*) $y = x - 1$ and $y = x + 4$

(*c*) $y = 2x + 3$ and $y = 2x + 5$ (*d*) $y = 3x + 11$ and $3y + x + 2 = 0$

7.6 GRAPHS OF QUADRATIC FUNCTIONS

Any function of the form $y = ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$, is called a *quadratic function* or *relation*. Thus a quadratic function is one containing x^2 as well as (perhaps) x and a number. $y = 3x^2 + 5x - 2$ is an example of a quadratic relation. The following illustrative examples will help us to draw quadratic graph.

The procedure is the same has been used for drawing linear graphs in the previous section.

Example 12. (*a*) *Copy and complete the table of values below for the relation*

$$y = x^2 - 2x - 3 \text{ for } -2 \leq x \leq 4$$

x	–2	–1	0	1	2	3	4
y	5		–3			0	

(*b*) *Using a scale of 2 cm to 1 unit on both the axes draw the graph of the relation.*

(*c*) *From your graph, find*

(*i*) *the positive value of y when $x = -1.5$*

(*ii*) *the positive value of x when $y = -1.7$*

Solution. (*a*) Given $y = x^2 - 2x - 3$, we can complete the table by substituting the given values of x in the relation to obtain their corresponding y values.

Note that the values of y for $x = -2$, 0 and 3 have already been given.

For $x = -1$, $y = (-1)^2 - 2(-1) - 3 = 1 + 2 - 3 = 0$

For $x = 1$, $y = (1)^2 - 2(1) - 3 = 1 - 2 - 3 = -4$

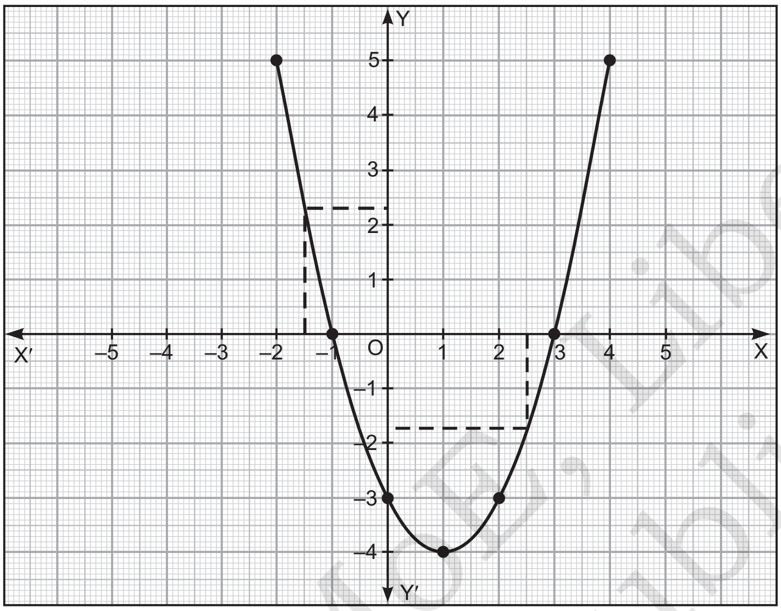
For $x = 2$, $y = (2)^2 - 2(2) - 3 = 4 - 4 - 3 = -3$

For $x = 4$, $y = (4)^2 - 2(4) - 3 = 16 - 8 - 3 = 5$

Hence the completed table is shown below:

x	–2	–1	0	1	2	3	4
y	5	0	–3	–4	–3	0	5

(*b*) The graph is shown in the figure.



(*c*) (*i*) From the graph, the positive value of y when $x = -1.5$ is **2.3.**

(*ii*) Also, the positive value of x when $y = -1.7$ is **2.5.**

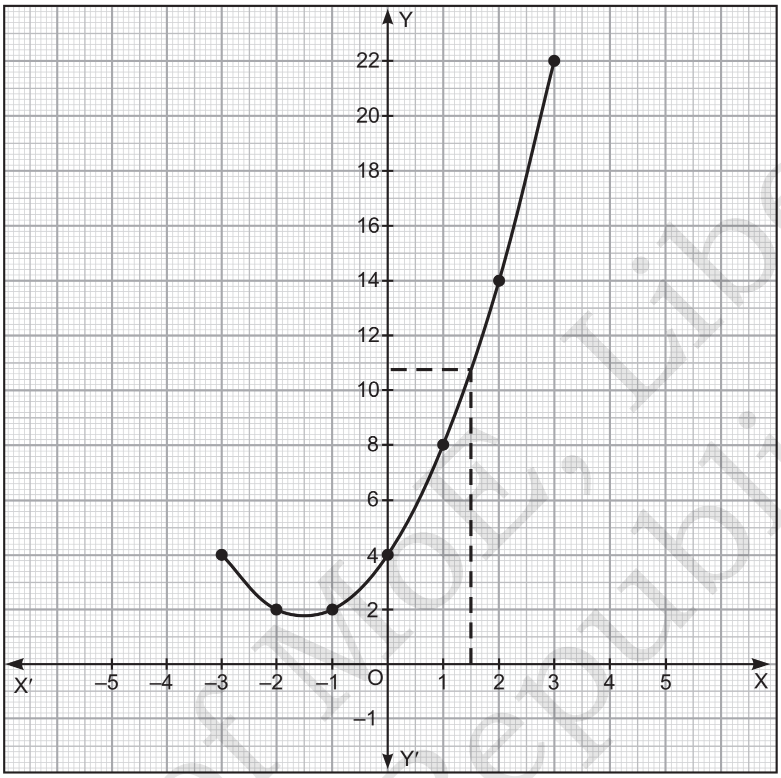
Example 13. (*a*) *Draw the graph of the relation* $y = x^2 + 3x + 4$ *for the interval* $-3 \le x \le 3$.

(*b*) *Use your graph to find the value of y when* $x = 1.5$.

Solution. Note that in this case an incomplete table is not given, therefore it is easier to set out your calculations as follows:

x	–3	–2	–1	0	1	2	3
x^2	9	4	1	0	1	4	9
$3x$	–9	–6	–3	0	3	6	9
4	4	4	4	4	4	4	4
y	4	2	2	4	8	14	22

The graph is shown in the figure.



(*b*) From the graph the value of y when x = 1.5 is **10.6.**

EXERCISE 7.6

1. Draw the graph of the function $y = x^2 + 2x$ from $x = -2$ to $x = 4$. Use your graph to find the values of y when x is 2.5.

2. The table below is a table of values for the function $y = x^2 - 6x + 5$ from $x = 0$ to $x = 6$.

x	0	1	2	3	4	5	6
y	5	0	–3	–4	–3	0	5

Draw the graph of the given function using the above table.
Use your graph to find the values of y when x is 1.5.

3. The table below shows values for the function $y = 12 - x - x^2$ from $x = -4$ to $x = 4$. Use your graph to find the value of x when y is 4.

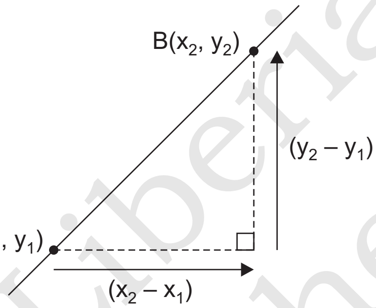
x	–4	–3	–2	–1	0	1	2	3	4
y	–8	6	10	12	12	10	6	0	–8

7.7 GRADIENT OF A STRAIGHT LINE

Joining Two Given Points

Let A(x_1, y_1) and B(x_2, y_2) be any two points on a straight line as shown in the figure.

The *gradient or slope* of the line joining points A and B is given by:



$$m = \frac{\text{Vertical distance}}{\text{Horizontal distance}}$$

$$= \frac{\text{Difference in } y\text{-values}}{\text{Difference in } x\text{-values}}$$

$$= \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{\Delta y}{\Delta x}$$

Note: The order of subtraction is very important.

Example 14. *Find the gradient of the line which passes through the points:*

(*a*) *A(1, 1) and B(7, 2)* (*b*) *P(2, 1) and Q(5, 5)*
(*c*) *L(3, – 2) and M(– 3, 4).*

Solution. (*a*) The gradient of line AB = $\frac{2-1}{7-1} = \frac{1-2}{1-7} = \frac{1}{6}$.

(*b*) The gradient of line PQ = $\frac{5-1}{5-2} = \frac{1-5}{2-5} = \frac{4}{3}$

(*c*) The gradient of line LM = $\frac{4-(-2)}{-3-3} = \frac{6}{-6} = -1$.

Finding Gradient of a Straight Line When its Equation is Given

To find the gradient, express the given equation in the form $y = mx + c$ and take the coefficient of x (*i.e.* m) as the gradient.

$$y = mx + c$$

Gradient (m) Intercept on y-axis (c)

Example 15. *Find the gradient of the line with equation 2x + 4y = 7.*

Solution. Making y the subject of the equation,

$$2x + 4y = 7 \quad \Rightarrow \quad 4y = -\,2x + 7$$

$$\Rightarrow \qquad y = -\frac{1}{2}x + \frac{7}{4}$$

The coefficient of x is $-\frac{1}{2}$.

$\therefore$ The gradient of the line is $-\frac{1}{2}$.

EXERCISE 7.7

1. Find the gradient of the line which passes through the points:
(*a*) A(3, 1) and B(6, 10) (*b*) X(5, – 1) and Y(3, 5)
(*c*) P(3, – 2) and Q(6, 7)

2. Find the gradient of the lines passing through the following pairs of points;
(*a*) (0, 0) and (1, 3) (*b*) (1, 4) and (3, 7)
(*c*) (5, 4) and (2, 3)

3. Find the gradient of the following straight lines.
(*a*) $x + y = 4$ (*b*) $2x + y = 3$

4. Find the gradient of the line with equations:
(*a*) $3x + y - 2 = 0$ (*b*) $2x - 4y - 1 = 0$

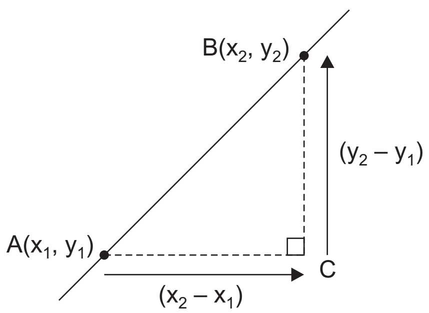
7.8 DISTANCE BETWEEN TWO POINTS

Let A(x_1, y_1) and B(x_2, y_2) be any two points on a straight line as shown in the figure.

Using Pythagoras theorem,

$$|AB^2| = |AC^2| + |CB^2|$$

$$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Therefore the length of the straight line joining the two points A and B is given by:

$$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(\Delta x)^2 + (\Delta y)^2}\,.$$

Example 16. *Find the length of the line joining the points P(4, 3) and Q(4, 7).*

Solution. $|PQ| = \sqrt{(4-4)^2 + (7-3)^2}$

$= \sqrt{(0)^2 + (4)^2} = \sqrt{16} = 4$ units

Example 17. *What is the distance between the points A(5, – 6) and B(2, 5)?*

Solution. $|AB| = \sqrt{[(5-(2)]^2 + [(-6-5)]^2}$

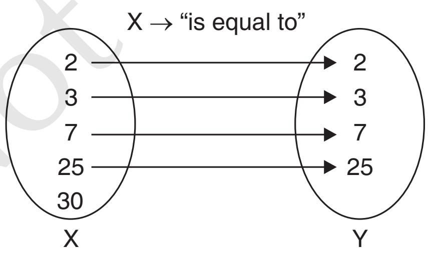
$= \sqrt{(3)^2 + (-11)^2} = \sqrt{9+121} = \sqrt{130}$ units.

EXERCISE 7.8

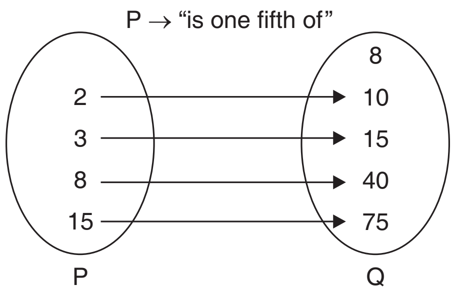
1. Find the length of the lines passing through the following pairs of points.
   (*a*) (1, 2) and (4, 6) (*b*) (3, 1) and (2, 0)
   (*c*) (4, 3) and (5, 2) (*d*) (7, 2) and (3, 6)
   (*e*) (0, 0) and (–1, –2).
2. Find the length of the line joining the following points:
   (*a*) P(1, 3) and Q(–2, 7) (*b*) A(–2, –2) and B(7, 10)
   (*c*) A(–4, 3) and B(5, 16) (*d*) C(–1, 2) and D(5, –6)

REVIEW EXERCISE

1. Identify and write the relation between the following pairs of sets:
   (*a*) 26th July .... Liberia (*b*) 4 16
   (*c*) 11 .... 20 (*d*) 100 50
   (*e*) 2^6 64
2. The following arrow diagram shows a relation between the two sets X and Y.
   Find the domain, co-domain and range.



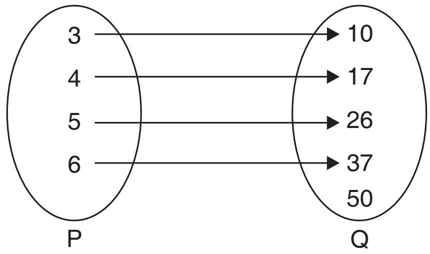
Q.2



Q.3

3. Find the domain, co-domain and range of the following relation.

4. Find the domain, co-domain and range of the following relation:



Q.4

5. Find the rule of the mapping: {(–1, 3), (–2, 6), (–3, 9), (–4, 12)}

6. Find the value of x which makes the following relations not defined

(*a*) $x \to \dfrac{x+1}{x^2 - 10x}$ (*b*) $x \to \dfrac{5x^2}{x^3 - x^2}$

7. Let $f: x \to 2x - 3,\ x \in \{-2, -1, 0, 1, 2, 3\}$. Find the range of f.

8. Given that $f(x) = px + q$, find the values of p and q, if $f(2) = 4$ and $f(4) = 10$.

9. The function f and g are defined as follows: $f: x \to \dfrac{x-1}{2}$ and $g: x \to 3x + 1$. (*a*) Evaluate $f\left(-\dfrac{1}{2}\right) + 1$ (*b*) Solve $f(x) = g(-2)$.

10. Change the subject of each of the following formulas to the letter given against them:

(*a*) $S = \dfrac{n}{2}(a + l),\ l$ (*b*) $S = 2\pi r(r + h),\ h$

(*c*) $S = ut + \dfrac{1}{2}at^2,\ a$ (*d*) $l = a + (n - 1)d,\ n$

11. Draw the graph of the following pair of linear functions.

(*a*) $x = 2$ and $y = -1$ (*b*) $y = x + 2$ and $y = x$

12. Draw the graph of the function $y = 2x^2 + 3x - 7$ from $x = -2$ to $x = 4$.

13. Find the gradient of the lines passing through the following pairs of points;

(*a*) (2, 5) and (5, 9) (*b*) (–1, 2) and (2, –3) (*c*) (1, –3) and (5, 3)

14. Find the gradient of the line with equations:

(*a*) $5x - y + 3 = 0$ (*b*) $x + 7y - 5 = 0$

15. Find the length of the line joining the following points A(–5, 3) and B(5, 9).

16. Make d the subject of the formula $S = \dfrac{n}{2}[2a + (n - 1)d]$.

MULTIPLE CHOICE QUESTIONS (MCQs)

1. If $x \to 3x - 4$, what is the image of –2?

(a) –10 (b) 2 (c) –2 (d) –1

2. What is the image of 3 in the mapping $x \to 3x + 7$

(a) 24 (b) 10 (c) 16 (d) 13

3. Find the image of –3 under the mapping $x \to 2(x + 3)$

(a) 0 (b) 12 (c) 2 (d) 6

4. What is the rule for the mappling below.

x	0	1	2	3	4
↓	↓	↓	↓	↓	↓
y	0	1	4	9	16

(a) $x \to \sqrt{x}$ (b) $x \to x + 2$ (c) $x \to x^2$ (d) $x \to x + 1$

5. Find the rule for the following mapping:

x	0	1	2	3	4
↓	↓	↓	↓	↓	↓
y	–1	1	3	5	7

(a) $y = x^2 + 1$ (b) $y = 2x - 1$ (c) $y = 2x + 1$ (d) $y = x^2 - 1$

6. Find the rule for the mapping:

1	2	3	4	5	n
↓	↓	↓	↓	↓	↓
10	21	32	43	54	…

(a) $n \to (10n + 1)$ (b) $n \to 10n$ (c) $n \to (11n - 1)$ (d) $n \to (7n + 3)$

7. Write the rule for the mapping

x	1	2	3	4
↓	↓	↓	↓	↓
y	1	3	5	7

(a) $x \to 2(x - 1)$ (b) $x \to 2x + 1$ (c) $x \to 2(x + 1)$ (d) $x \to 2x - 1$

8. If $(x, 7) = (5, 7)$, then the value of x is:

(a) 5 (b) 7 (c) –5 (d) None of these

9. The gradient of the line passing through the points (–5, –3) and (4, 7) is

(a) $-\frac{10}{9}$ (b) $\frac{10}{9}$ (c) $\frac{9}{10}$ (d) $-\frac{9}{10}$

10. The gradient of the line of equation $7x - 5y + 8 = 0$ is

(a) $\frac{7}{5}$ (b) $-\frac{7}{5}$ (c) $\frac{5}{7}$ (d) $-\frac{5}{7}$

11. The domain and the range of the real function $f(x) = \frac{4-x}{x-4}$ is given by

(a) Domain = **R**, Range = {–1, 1} (b) Domain = **R** – {1}, Range = **R**
(c) Domain = **R** – {4}, Range = {–1} (d) Domain = **R** – {–4}, Range = {– 1, 1}

12. The length of the line joining the points (7, 4) and (–3, –1) is:

(a) $5\sqrt{2}$ units (b) $5\sqrt{3}$ units (c) $5\sqrt{5}$ units (d) $5\sqrt{7}$ units

RECAP AT A GLANCE

- A relation can be represented by matching diagram.
- Domain of a relation is the set of all elements in the *first set* from the direction of the arrow diagram.
- Co-domain of a relation is the set of elements in the *second set* from the direction of the arrow diagram.
- The range is a subset of the co-domain and denoted by R.
- An *ordered pair* is a pair of objects taken in a specific order.
- A *function* is a mapping in which each element in the domain is mapped onto one and only one member in the co-domain.
- *Subject of a formula* is the variable which is expressed in terms of other variables involved in the formula.

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PERIOD-V

M10CH8

TOPIC 8 Simultaneous Linear Equations

8.1 SIMULTANEOUS LINEAR EQUATIONS

Simultaneous linear equations are a pair of equations which are true (solved) at the same time i.e. simultaneously. If we have two linear equations and we wish to make both equations true at the same time, we require values for the variables which satisfy both the equations. These values are the *simaltaneous solutions* to the pair of equations.

Consider a pair of *simultaneous linear equations* containing two unknowns, usually x and y. There are infinitely many values of x and y which satisfy the *first equation*. Likewise, there are infinitely many values of x and y which satisfy the *second equation*. In general, however, only one combination of values of x and y satisfies *both* the equations at the same time.

For example: Consider the pair of simultaneous linear equations

$$\begin{cases} x + y = 9 \\ 2x + 3y = 21 \end{cases}$$

If $x = 6$ and $y = 3$ then

- $x + y = 6 + 3 = 9$. The first equation is satisfied.
- $2x + 3y = 2(6) + 3(3) = 12 + 9 = 21$. The second equation is also satisfied.

So, $x = 6$ and $y = 3$ is the *solution* to the simultaneous equations

$$x + y = 9 \quad \text{and} \quad 2x + 3y = 21.$$

Example 1. *Find the simultaneous solution to the equations:*

$$y = 2x - 1 \quad \text{and} \quad y = x + 3.$$

Solution. If $y = 2x - 1$ and $y = x + 3$, then

$$2x - 1 = x + 3 \qquad \text{(Equating } y\text{'s)}$$

$$\therefore \quad 2x - 1 - x = x + 3 - x \qquad \text{(Subtracting } x \text{ from both sides)}$$

$\therefore \quad x - 1 = 3$

$\therefore \quad x = 4$ (Adding 1 to both sides)

and so $\quad y = 4 + 3$ (Using $y = x + 3$)

$\therefore \quad y = 7$

So, the simultaneous solution is $x = 4$ and $y = 7$.

Check: In $y = 2x - 1$, $\quad y = 2 \times 4 - 1 = 8 - 1 = 7$

In $y = x + 3$, $\quad y = 4 + 3 = 7$

8.2 TRUTH SET FOR SIMULTANEOUS LINEAR RELATIONS

Solving an equation is the process of finding the truth values of the equation. The set of all truth values of an equation is the truth set (solution set) of the equation.

Example 2. *Consider the following simultaneous linear equations and find their common set of truth values: $2y - x = 8$ and $y - 2x = 1$.*

Solution. We have $\quad 2y - x = 8$

By subject changing formula, we get

Arbitrary values $\quad x = 2y - 8$ ...(1)

When $y = 2$, $\quad x = 2(2) - 8 = 4 - 8 = -4$

When $y = 3$, $\quad x = 2(3) - 8 = 6 - 8 = -2$ (Fixed values)

When $y = 5$, $\quad x = 2(5) - 8 = 10 - 8 = 2$

and so on.

$\therefore$ Truth set of equation (1) = {(–4, 2), (–2, 3), (2, 5)} = S_1 (say)

Similarly, for $y - 2x = 1$ ...(2)

Arbitrary values $\quad y = 1 + 2x$

When $x = -1$, $\quad y = 1 + 2(-1) = 1 - 2 = -1$

When $x = 0$, $\quad y = 1 + 2(0) = 1$ (Fixed values)

When $x = 2$, $\quad y = 1 + 2(2) = 1 + 4 = 5$

and so on.

$\therefore$ Truth set of equation (2) = {(–1, –1), (0, 1), (2, 5)} = S_2 (say)

Now, $\quad S_1 \cap S_2 = \{(2, 5)\}$ (Singleton set)

Which is the common set of truth values of the given simultaneous equations.

Example 3. *Find the truth set of the following equations: $x + y = 3$ and $3x - 2y = 4$. Also find their common set of truth values.*

Solution. For $x + y = 3$

By subject changing formula, we get

Arbitrary values

$y = 3 - x$ ...(1)

When $x = 1$, $y = 3 - 1 = 2$
When $x = 2$, $y = 3 - 2 = 1$ } Fixed values
When $x = 3$, $y = 3 - 3 = 0$

and so on

$\therefore$ Truth set of equation (1),

$$S_1 = \{(1, 2), (2, 1), (3, 0)\}$$

Similarly, for $3x - 2y = 4$, we have

$$2y = 3x - 4 \Rightarrow y = \frac{3x-4}{2} \quad ...(2)$$

Arbitrary values

When $x = 0$, $y = \frac{3(0)-4}{2} = \frac{-4}{2} = -2$

When $x = 4$, $y = \frac{3(4)-4}{2} = \frac{8}{2} = 4$ } Fixed values

When $x = 2$, $y = \frac{3(2)-4}{2} = \frac{6-4}{2} = \frac{2}{2} = 1$

and so on.

Note: You may find more truth set but you can stop when you find out the common set.

$\therefore$ Truth set of equation (2), $S_2 = \{(0, -2), (4, 4), (2, 1)\}$

Now, $S_1 \cap S_2 = \{(2, 1)\}$, which is the required truth set that is singleton.

EXERCISE 8.1

1. Find the simultaneous solution to the following pairs of equations:
   (*a*) $y = x - 2$, $y = 3x + 6$ (*b*) $y = x + 2$, $y = 2x - 3$ (*c*) $y = 6x - 6$, $y = x + 4$ (*d*) $y = 2x + 1$, $y = x - 3$
2. Consider the following simultaneous linear equations and find their common set of truth values.
   (*a*) $x + 3y = 6$; $2x - 3y = 12$ (*b*) $2x - y - 4 = 0$; $x + y + 1 = 0$
   (*c*) $2x - y = 2$; $4x + 3y = 24$

8.3 FINDING SOLUTION(S) USING GRAPHS

Two linear equations in two variables form *a system of linear equations.* A *solution* to a system of linear equations is an ordered pair which satisfies both equations in the system.

For example: The ordered pair (3, 4) satisfies the system of equations

	$2x - 3y + 6 = 0$	and	$2x - y - 2 = 0$
i.e.,	$2 \times 3 - 3 \times 4 + 6 = 0$	*i.e.,*	$2 \times 3 - 4 - 2 = 0$
or	$6 - 12 + 6 = 0$	or	$6 - 6 = 0$

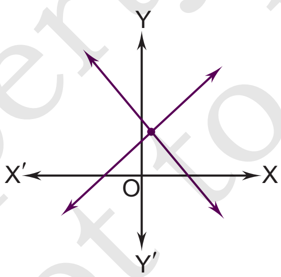
both are true statements.

Thus, $x = 3$ and $y = 4$ or the ordered pair (3, 4) is a solution. The solution set to the system is {(3, 4)}.

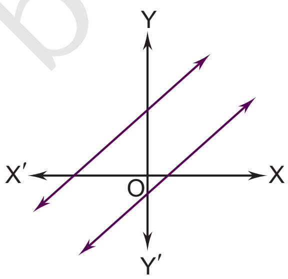
The solution can be obtained by graphing both equations. The coordinates of the point of intersection give the solution set of the system.

If the lines intersect, they will intersect in only one point giving a unique solution to the system as shown in the figure (*i*) If the lines are parallel, there is no point of intersection and, hence, no solution (see figure (*ii*)).

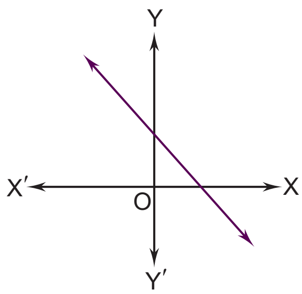
If the lines are coincident, *i.e.,* the same line for both equations, then every point on the line is a common point and, hence, there are infinitely many solutions (see figure (*iii*)).



(i) Unique solution (intersecting lines)



(ii) No solution (parallel lines)



(iii) Infinitely many solutions (coincident lines)

We shall now concentrate on the system having a unique solution.

Example 4. *Find the solution set of the following system of equations graphically:* $2x + 5y = 10$ *and* $x = -5$.

Solution. Given equations are:

$$2x + 5y = 10 \quad \Rightarrow \quad y = \frac{10 - 2x}{5} \quad \ldots(1) \qquad x = -5 \quad \ldots(2)$$

Table of values for (1)

x	0	5	→ Arbitrary
y	2	0	→ Fixed

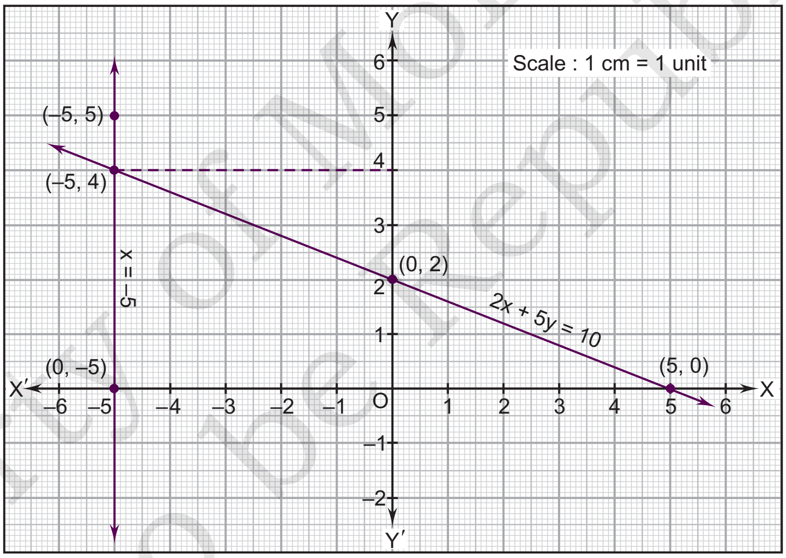
Table of values for (2)

x	–5	–5	→ Arbitrary
y	0	5	→ Fixed

Plot the ordered pairs (0, 2) and (5, 0). Join and produce both ways. This is the graph of equation (1).

Plot the ordered pairs (–5, 0) and (–5, 5). Join and produce both ways. This vertical line is the graph of equation (2).

The two lines intersect at a unique point (–5, 4) (as shown in the figure). Therefore, the ordered pair (–5, 4), *i.e.*, $x = -5$, $y = 4$ is the solution of the system. The solution set $S = \{(-5, 4)\}$ is a singleton set.



Example 5. *Find the solution set of the following system of equations graphically:* $2x - y - 1 = 0$ *and* $x - 2y + 1 = 0$.

Solution. The given equations can be written as

$$y = 2x - 1 \quad \ldots(1) \qquad y = \frac{x + 1}{2} \quad \ldots(2)$$

Table of values for (1)

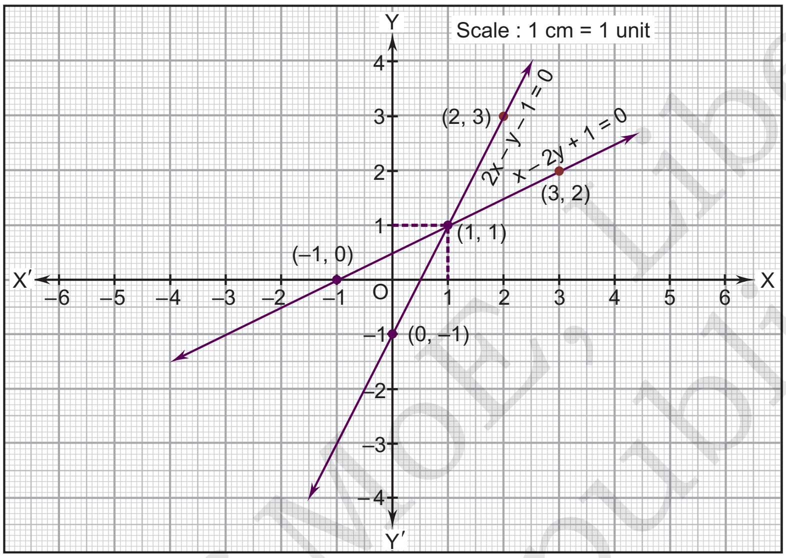
x	0	2
y	–1	3

Table of values for (2)

x	–1	3
y	0	2

Plot the ordered pairs (0, –1) and (2, 3). Join and produce both ways. This is the graph of equation (1).

Plot the ordered pairs (–1, 0) and (3, 2). Join and produce both ways. This is the graph of equation (2) (see the figure given below).



The two lines intersect at a unique point (1, 1). Therefore, the solution set is S = {(1, 1)}.

EXERCISE 8.2

1. Find the solution set of the following system of equations graphically:
$2x - 3y = 1$, $3x - 4y = 1$.

2. Solve the following pair of linear equations graphically:
$2x + y - 6 = 0$, $4x - 2y - 4 = 0$.

3. Draw the graph of the equations $x - y + 1 = 0$ and $3x + 2y - 12 = 0$. Using this graph, find the values of x and y which satisfy both the equations.

8.4 ELIMINATION METHOD

Under this method, we seek to make one of the two variables becomes *zero* and then solve for the *other variable* as in linear equation in one variable. Note the following steps.

1. Make the coefficient of the variable you wish to eliminate the same (*i.e.* to have the same absolute value).

2. If the values of both coefficients are positive or negative, then *subtract*, but if one is positive and the other is negative, then *add* the equations.
3. Solve the resulting equation and find the value of its variable.
4. Substitute (or replace) the value of the variable obtained in *step 3* above into one of the original equations and solve for the remaining variable.

Note: If the equations are mixed up, re-arrange them before solving.

Example 6. *Solve the following simultaneous equations:*

(*i*) $x + y = 7,\ 2x - y = 5$ (*ii*) $\frac{1}{2}a + \frac{1}{3}b = 1,\ \frac{1}{4}a - \frac{1}{2}b = -2$

Solution.

(*i*)

$$x + y = 7 \quad \ldots(1)$$
$$+\ 2x - y = 5 \quad \ldots(2)$$
$$3x + 0y = 12$$

[Adding (1) and (2) in order to eliminate y]

$\Rightarrow$ $3x = 12 \Rightarrow x = \frac{12}{3} = 4$

Replacing $x = 4$ in (1), we have

$$4 + y = 7 \Rightarrow y = 7 - 4 = 3 \Rightarrow y = 3$$

$\therefore$ The solution is $x = 4$, $y = 3$.

(*ii*)

$$\frac{1}{2}a + \frac{1}{3}b = 1 \quad \ldots(1)$$
$$\frac{1}{4}a - \frac{1}{2}b = -2 \quad \ldots(2)$$

Multiplying (1) by the LCM(2, 3) = 6 and (2) by the LCM (2, 4) = 4,

(1) becomes $3a + 2b = 6$ ...(3)

(2) becomes $a - 2b = -8$ ...(4)

} Adding in order to eliminate y

$$4a = -2 \Rightarrow a = \frac{-2}{4} = -\frac{1}{2}$$

Replacing $a = -\frac{1}{2}$ in (1), $\frac{1}{2}\left(-\frac{1}{2}\right) + \frac{1}{3}b = 1$

$\Rightarrow$ $-\frac{1}{4} + \frac{1}{3}b = 1 \Rightarrow \frac{1}{3}b = 1 + \frac{1}{4}$

$\Rightarrow$ $\frac{1}{3}b = \frac{5}{4} \Rightarrow b = \frac{3 \times 5}{4} = \frac{15}{4} = 3\frac{3}{4}$

$\therefore$ The solution is $a = -\frac{1}{2}$ and $b = 3\frac{3}{4}$.

8.5 SUBSTITUTION METHOD

To solve a pair of simultaneous linear equations by the substitution method, the following steps should be taken.

1. Make one of the variables the subject in one of the equations and substitute it into the *other* equation.
2. Solve for the variable in the equation obtained in step one.
3. Substitute the value of the variable you have solved for in step two above into one of the original equations and solve for the other variables.

Example 7. *Solve the following system of equations:*

(*i*) $x + 3y = 8, x + y = 2$ (*ii*) $5x - 3y = 30, x = -3y - 12$

Solution. (*i*) $x + 3y = 8$ ...(1)

$x + y = 2$ ...(2)

Making x the subject in (2) and substituting it in (1),

(2) becomes $x = 2 - y$

Substituting $x = 2 - y$ in (1),

$2 - y + 3y = 8 \Rightarrow 2y = 8 - 2$

$\Rightarrow 2y = 6 \Rightarrow y = 3$

Substituting $y = 3$ in (2),

$\Rightarrow x + 3 = 2 \Rightarrow x = 2 - 3$

$\therefore x = -1$

Therefore the solution is: $x = -1,\ y = 3$.

(*ii*) $5x - 3y = 30$ ...(1)

$x = -3y - 12$ ...(2)

Substituting equation (2) in equation (1),

$5(-3y - 12) - 3y = 30$

$\Rightarrow -15y - 60 - 3y = 30 \Rightarrow -18y = 90$

$\Rightarrow \frac{-18y}{-18} = \frac{90}{-18} \Rightarrow y = -5$

Substituting $y = -5$ in equation (2),

$x = -3(-5) - 12 \Rightarrow x = 15 - 12$

$\therefore x = 3$

Therefore the solution is: $x = 3,\ y = -5$.

EXERCISE 8.3

1. Solve each of the following pairs of linear equations by the *Elimination Method:*

(*a*) $2x - y = 6;\ x - y = 2$ (*b*) $x + y = 6;\ x - y = 2$

(*c*) $7x - 8y - 11 = 0;\ 8x - 7y - 7 = 0$ (*d*) $2x + 7y - 11 = 0;\ 3x - y - 5 = 0$

2. Solve each of the following pairs of linear equations by the *Elimination Method*:

(*a*) $4x + \frac{y}{3} = \frac{8}{3};\ \frac{x}{2} + \frac{3y}{4} = -\frac{5}{2}$ (*b*) $\frac{3}{x} - 5y + 1 = 0;\ \frac{2}{x} - y + 3 = 0$

(*c*) $\frac{4}{x} + 3y = 14;\ \frac{3}{x} - 4y = 23.$

3. Solve each of the following pairs of linear equations by the *Elimination Method:*

(*a*) $47x + 31y = 63;\ 31x + 47y = 15$

(*b*) $37x + 41y = 70;\ 41x + 37y = 86.$

4. Solve the following pair of linear equations for x and y, by the *Substitution Method*:

(*a*) $7x - 15y = 2;\ x + 2y = 3$ (*b*) $2x + 3y = 9;\ 3x + 4y = 5$

(*c*) $x - y = 0;\ 2x - y = 2$ (*d*) $2x - y = 2;\ 3x - 4y = -2$

5. Solve for x and y by the *Substitution Method*:

(*a*) $2x = 5y + 4;\ 3x - 2y + 16 = 0$ (*b*) $2x - y + 3 = 0;\ 3x - 5y + 1 = 0$

8.6 WORD PROBLEMS ON SIMULTANEOUS LINEAR EQUATIONS

To solve a word problem involving two unknown quantities:

(*i*) Assume the two unknown quantities as x and y.

(*ii*) Using given conditions construct two linear equations in two variables x and y.

(*iii*) Solve the pair of linear equations simultaneously to get the values of x and y.

Example 8. *A family of three adults and two children paid L$ 72 as the fare for a journey. Another family of four adults and three children paid L$ 99 as the fare for the same journey. Calculate the fare for*

(*i*) *an adult* (*ii*) *a child*

Solution. Let the fare for an adult be L$ x and for a child be L$ y.

Given:

(*i*) Fare for three adults (= L$ $3x$) and two children (= L$ $2y$) is L$ 72

$\Rightarrow \quad 3x + 2y = 72 \quad$ ...(1)

(*ii*) Fare for four adults (= L$ $4x$) and three children (= L$ $3y$) is L$ 99

$\Rightarrow \quad 4x + 3y = 99 \quad$ ...(2)

To make the coefficient of y equal in the two equations, we multiply (1) by 3 and (2) by 2, getting

$9x + 6y = 216 \quad$ ...(3)

and $\quad 8x + 6y = 198 \quad$ ...(4)

Subtracting (4) from (3), we get $x = 18$

Replacing x by 18 in equation (1), we get

$$3 \times 18 + 2y = 72 \Rightarrow 2y = 72 - 54$$

$$\Rightarrow \quad 2y = 18 \Rightarrow y = 9$$

Therefore,

(*i*) the fare for an adult (x) = L$ 18

(*ii*) the fare for a child (y) = L$ 9

Example 9. *A number consists of two digits whose sum is 8. When 18 is added to the number, the digits are reversed. Find the number.*

Solution. Let x be the digit in ten's place and y be the digit in unit's place. Then the number is $10x + y$. When the digits are reversed, we have y in ten's place and x in unit's place so that the new number is $10y + x$.

Given: (*i*) Sum of digits is 8

$\Rightarrow \quad x + y = 8 \quad$ ...(1)

(*ii*) Original number + 18 = New number

$$\Rightarrow \quad (10x + y) + 18 = 10y + x$$

$$\Rightarrow \quad 10x - x + y - 10y = -18$$

$$\Rightarrow \quad 9x - 9y = -18$$

Dividing throughout by 9, we have

$x - y = -2 \quad$ ...(2)

Adding (1) and (2), we get

$$2x = 6 \quad \Rightarrow \quad x = \frac{6}{2} = 3$$

Replacing x by 3 in (1), we have

$$3 + y = 8 \Rightarrow y = 8 - 3 \Rightarrow y = 5$$

Therefore, the required number is 35.

Example 10. *Five years ago, father's age was six times his son's age. Five years hence, their ages will be in the ratio 8 : 3. Find their present ages.*

Solution. Let father's present age be x years and son's present age be y years.

Five years ago, their ages were $(x - 5)$ years and $(y - 5)$ years.

Given: $x - 5 = 6(y - 5) \Rightarrow x - 5 = 6y - 30$

$\Rightarrow x - 6y - 5 + 30 = 0 \Rightarrow x - 6y + 25 = 0$ ...(1)

Five years hence, their ages will be $(x + 5)$ years and $(y + 5)$ years.

Given: $\dfrac{x+5}{y+5} = \dfrac{8}{3} \Rightarrow 3(x + 5) = 8(y + 5)$

$\Rightarrow 3x + 15 = 8y + 40 \Rightarrow 3x - 8y + 15 - 40 = 0$

$\Rightarrow 3x - 8y - 25 = 0$ ...(2)

Multiplying (1) by 3, we have

$3x - 18y + 75 = 0$ ...(3)

Subtracting (3) from (2),

$$10y - 100 = 0 \quad \text{or} \quad 10y = 100$$

$$\Rightarrow \quad y = \frac{100}{10} = 10$$

Putting $y = 10$ in (2), we have

$3x - 8 \times 10 - 25 = 0$

$\Rightarrow 3x - 80 - 25 = 0 \Rightarrow 3x - 105 = 0$

$\Rightarrow 3x = 105 \Rightarrow x = \dfrac{105}{3} = 35$

Hence, father's present age = 35 years and son's present age = 10 years.

EXERCISE 8.4

1. The sum of two numbers is 29. Their difference is 17. Find the two numbers.
2. One pencil and 2 erasers cost L$ 8. Two pencils and 3 erasers cost L$ 14. How much does each pencil and each eraser cost?

3. The sum of the ages of Emine and Samuel is 29 years. Emine is 3 years older than Samuel. How old are they?
4. Tickets for a film were sold L$ 450 to the general public and L$ 375 to pupils. 400 people attended the show and L$ 168000 was collected in ticket sales.
   (*a*) How many tickets were sold to pupils?
   (*b*) Mr. Samuel was issued with 25 tickets to be sold to the general public and 20 tickets to be sold to pupils. How much did Mr. Samuel collect after selling the tickets issued to him?
5. The sum of two numbers is 8 and their product is –33. Find the two numbers.
6. The total age of two sisters is 108 years. One is 18 years older than the other. Find the ratio of the age of the older to the younger.

REVIEW EXERCISE

1. Find the simultaneous solution of the following pairs of equations:
   (*a*) $y = x + 4$, $y = 5 - x$ (*b*) $y = x + 1$, $y = 7 - x$ (*c*) $y = 2x - 5$, $y = 3 - x$ (*d*) $y = x - 4$, $y = -2x - 4$
2. Consider the following simultaneous linear equations and find their common set of truth values.
   (*a*) $2x + y + 6 = 0$; $2x - y + 2 = 0$ (*b*) $x - y = 1$; $2x + y = 8$
3. Find the solution set of the following system of equations graphically:
   $3x - 4y + 6 = 0, \quad 3x + y - 9 = 0.$
4. Solve the following pair of linear equations graphically:
   $x + 3y = 6; 2x - 3y = 12.$
5. Solve each of the following pairs of linear equations by the *Elimination Method*:
   (*a*) $\frac{4}{x} + 5y = 7; \frac{3}{x} + 4y = 5$ (*b*) $x + \frac{6}{y} = 6; 3x - \frac{8}{y} = 5$
6. Solve each of the following pairs of linear equations by the *Elimination Method:*
   (*a*) $31x + 13y = 57; 13x + 31y = 75$ (*b*) $37x + 43y = 123; 43x + 37y = 117$
7. Solve the following pair of linear equations for x and y, by the *Substitution Method*:
   (*a*) $3x - 5y = -1; x - y = -1$ (*b*) $x + 2y = -1; 2x - 3y = 12$
8. Solve for x and y by the *Substitution Method*:

$$3x + 4y = 13; 2x - 3y = 3$$

9. If the numerator is increased by 2 and the denominator is decreased by 3, a fraction equals one. If the numerator is decreased by 3 and the denominator increased by 4, the fraction becomes $\frac{1}{4}$. Find the fraction.

10. Mr. Daniel and Mr. Felix run a small business assembling two types of product. The cost of components and the labour needed for each product is shown in the table below.

	Cost of component	**Labour man-hours**
Type A	36	16
Type B	24	24

The business has L$ 156 available to buy components each week. The total labour available each week is 96 man-hours. How many products of each type can they assemble each week to maintain maximum production?

MULTIPLE CHOICE QUESTIONS (MCQs)

1. The system of a pair of simultaneous linear equations $x = 0$, $y = 3$ has

(a) no solution (b) a unique solution
(c) two solutions (d) infinitely many solutions

2. The pair of simultaneous equations $x = 4$ and $y = -3$ graphically represent lines which are

(a) coincident (b) parallel
(c) intersecting at (4, –3) (d) intersecting at (–3, 4)

3. If $2x + 3y = 13$ and $5x - 4y = -2$, then $x + y$ equals

(a) 6 (b) –6 (c) 5 (d) –5

4. The solution of the pair of simultaneous linear equations $29x + 37y - 103 = 0$, $37x + 29y - 95 = 0$, is

(a) $x = 1, y = 2$ (b) $x = 2, y = 1$ (c) $x = 2, y = 3$ (d) $x = 3, y = 2$

5. If the pair of simultaneous linear equations $2x - y - 3 = 0$, $2kx + 7y - 5 = 0$ has a unique solution $x = 1$, $y = -1$ then the value of k is

(a) 3 (b) 4 (c) 6 (d) –6

6. The pair of simultaneous linear equations $x + 2y - 5 = 0$, $7x + 3y - 13 = 0$ has a unique solution. Then, the values of x and y are

(a) $x = 1, y = 2$ (b) $x = 2, y = 1$ (c) $x = 3, y = 1$ (d) $x = 1, y = 3$

7. A number consists of two digits, whose sum is 10. If 18 is subtracted from the number, digits are reversed. Then the number is

(a) 73 (b) 64 (c) 55 (d) None of these

8. If the sum of the ages of a mother and her son in years is 65, and twice the difference their ages in years is 50, then the age of the mother is

(a) 45 years (b) 40 years (c) 50 years (d) None of these

RECAP AT A GLANCE

- Simultaneous linear equations are a pair of equations which are true (solved) at the same time i.e. simultaneously.
- Solving an equation is the process of finding the truth values of the equation.
- Two linear equations in two variables form *a system of linear equations.*
- If the equations are mixed up, re-arrange them before solving.

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PERIOD-V

M10CH9

TOPIC

9

Vector in a Plane

9.1 SCALAR *AND* VECTOR QUANTITIES

Quantities are of two types, namely scalars and vectors.

Scalars

A quantity that has only magnitude is called a *scalar.*

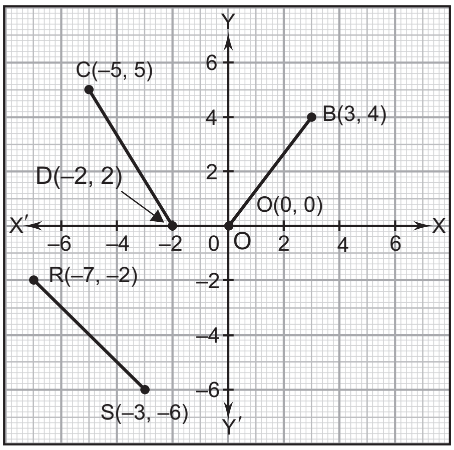
For example: Mass, length, time, temperature, area, volume, speed, density etc. are scalars.

Vectors

A quantity that has magnitude as well as a direction is called a *vector.*

For example: Displacement, velocity, acceleration, force etc. are vectors.

Representation of Vectors



A *vector* is any physical quantity that has *length* (or magnitude) and *direction.* In other words, vectors represent the length and directions of lines in the number plane.

For example: $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$,

$\overrightarrow{CD} = \begin{pmatrix} -5 \\ 6 \end{pmatrix}$, $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ etc.

Notice that vectors are expressed in columns, one value on top of the other. The values at the top are called the *x-components* of the vectors, while the values at the bottom are called the *y-components*.

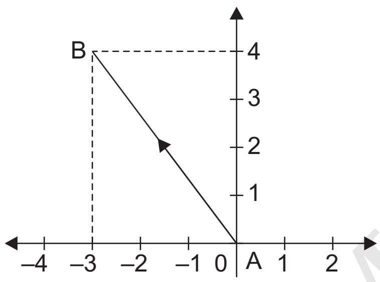
(See vectors $\overrightarrow{RS}$ and $\overrightarrow{CD}$ and $\overrightarrow{OB}$ in the figure). Note that the arrow placed on the vectors show the direction of the vectors.

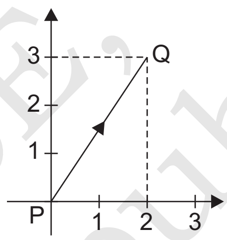
Example 1. *Express the following vectors graphically:*

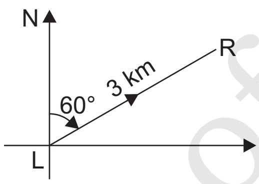
(*i*) $\overrightarrow{AB} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ (*ii*) $\overrightarrow{PQ} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

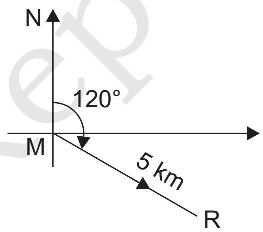
(*iii*) $\overrightarrow{LR}$ = (3 km, 060°) (*iv*) $\overrightarrow{MR}$ = (5 km, 120°)

Solution.

(*i*)


(*ii*)


(*iii*)


(*iv*)


EXERCISE 9.1

1. Express the following vectors graphically.

(*a*) $\overrightarrow{PQ}$ = (4 km, 30°) (*b*) $\overrightarrow{LM} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

2. Mark the following vectors on a graph sheet.

(*a*) $\overrightarrow{OP} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$; (*b*) $\overrightarrow{OQ} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ (*c*) $\overrightarrow{OR} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$; (*d*) $\overrightarrow{OS} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$

3. (*a*) Represent graphically a displacement of 40 km, 30° east of north.

(*b*) Represent graphically a displacement of 40 km, 30° west of south.

4. Classify the following as scalar and vectors:

(*a*) speed (*b*) velocity (*c*) mass (*d*) temperature
(*e*) acceleration (*f*) weight (*g*) time period (*h*) distance
(*i*) force (*j*) workdone.

9.2 TYPES OF VECTOR QUANTITIES

Zero Vector

A *zero vector* is a vector which *begins* and *ends* at the *same point.* Their length as well as direction is always zero.

For example: If P(2, 3) and Q(2, 3), then vector

$$\vec{PQ} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ i.e. } \vec{PQ} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Position Vector

A *position vector* is a vector which begins from the *origin* and ends at a point. $\vec{OB}$ is a vector see in the figure previous section 9.1, (*Page no. 170*).

For example: If P(4, 5), then $\vec{OP} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$ is the position vector. That is,

$$\vec{OP} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

Example 2. *Find $\vec{PQ}$ and hence $\vec{QP}$ if P(4, 8) and Q(–1, 5).*

Solution. The position vectors of P and Q referred to the origin O are $\vec{OP} = \mathbf{p} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$ and $\vec{OQ} = \mathbf{q} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$ respectively.

$$\therefore \quad \vec{PQ} = \vec{OQ} - \vec{OP} = \mathbf{p} - \mathbf{q}$$

$$= \begin{pmatrix} -1 \\ 5 \end{pmatrix} - \begin{pmatrix} 4 \\ 8 \end{pmatrix} = \begin{pmatrix} -5 \\ -3 \end{pmatrix}$$

$$\vec{QP} = -\vec{PQ} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}.$$

Negative Vector

Negative vector is the *opposite* or *inverse* of a vector.

For example: If $\vec{PQ} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ is a vector then vector $\vec{QP} = -\vec{PQ} = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$ is the negative or the opposite.

Example 3. (*i*) *If* $\vec{PQ} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$, *then find* $\vec{QP}$.

(*ii*) *If* $\vec{AB} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$, *then find* $\vec{BA}$.

Solution. (*i*) We have $\vec{PQ} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$

$\therefore$ Negative or opposite of $\vec{PQ}$ is $\vec{QP}$

$$\therefore \quad \vec{QP} = -\vec{PQ} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

(*ii*) We have $\vec{AB} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$

$\therefore$ Negative or inverse (opposite) of $\vec{AB}$ is $\vec{BA}$

$$\therefore \quad \vec{BA} = -\vec{AB} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}.$$

Equal Vectors

If any two or more vectors have the same components, then they are *equal vectors. For example:* $\vec{AB} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, $\vec{BC} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$ are equal vectors.

Example 4. If $\mathbf{p} = \begin{pmatrix} x+3 \\ y-1 \end{pmatrix}$ *and* $\mathbf{q} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$, *find x and y, if* $\mathbf{p} = \mathbf{q}$.

Solution. As $\mathbf{p} = \mathbf{q}$ (given) $\Rightarrow \begin{pmatrix} x+3 \\ y-1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

Equating corresponding components

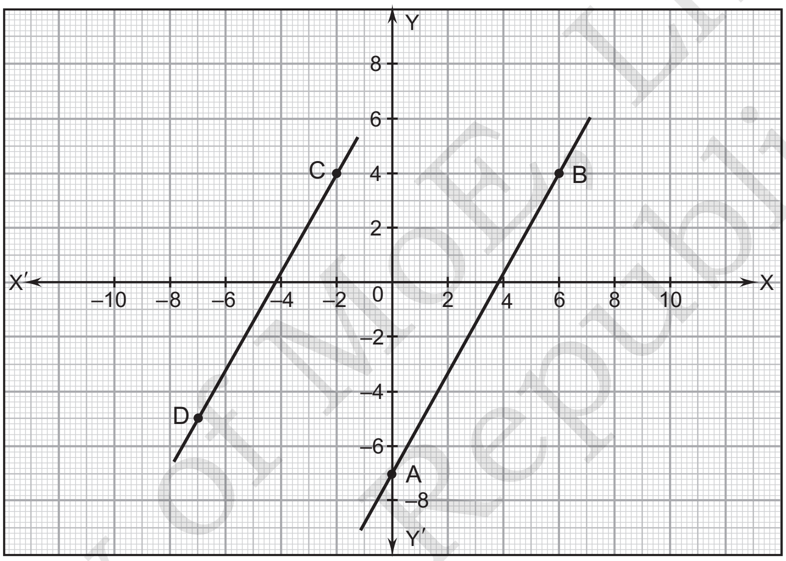
$\Rightarrow \quad x + 3 = -2 \quad \Rightarrow \quad x = -5$

and $\quad y - 1 = 3 \quad \Rightarrow \quad y = 4.$

Parallel Vectors

Two or more vectors are parallel, if one is a *scalar multiple* of the other(s).

For example: In the figure below, vector $\vec{AB}$ is parallel to vector $\vec{CD}$. Similarly, vector $\vec{AB} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ is parallel to vector $\vec{CD} = \begin{pmatrix} -3 \\ 9 \end{pmatrix}$ because $3\begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 9 \end{pmatrix}$.



Example 5. *Which of the following is parallel to* $\begin{pmatrix} 20 \\ 8 \end{pmatrix}$?

(*a*) $\begin{pmatrix} -25 \\ 10 \end{pmatrix}$ (*b*) $\begin{pmatrix} -2 \\ -5 \end{pmatrix}$ (*c*) $\begin{pmatrix} -15 \\ -6 \end{pmatrix}$ (*d*) $\begin{pmatrix} -4 \\ -10 \end{pmatrix}$ (*e*) $\begin{pmatrix} 4 \\ -10 \end{pmatrix}$

Solution. The vector which is parallel to $\begin{pmatrix} 20 \\ 8 \end{pmatrix}$ is a scalar multiple of $\begin{pmatrix} 20 \\ 8 \end{pmatrix}$. *i.e.* $k\begin{pmatrix} 20 \\ 8 \end{pmatrix}$ where k is a non-zero number. From the vectors given, $\begin{pmatrix} -15 \\ -6 \end{pmatrix}$ is parallel to $\begin{pmatrix} 20 \\ 8 \end{pmatrix}$ since $k = -\frac{3}{4}$.

Hence, the correct option is (*c*).

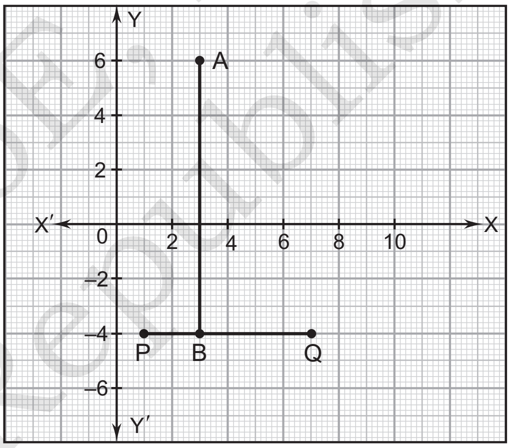
Note: In general, if the vector $\begin{pmatrix} a \\ b \end{pmatrix}$ is parallel to the vector $\begin{pmatrix} c \\ d \end{pmatrix}$, then $a : c = b : d$. That is the ratios of the corresponding components in the same order are equal.

Perpendicular Vectors

The vectors perpendicular to $\vec{AB} = \begin{pmatrix} a \\ b \end{pmatrix}$ are $\begin{pmatrix} b \\ -a \end{pmatrix}$ and $\begin{pmatrix} -b \\ a \end{pmatrix}$ or their scalar multiples $\begin{pmatrix} kb \\ -ka \end{pmatrix}$ and $\begin{pmatrix} -kb \\ ka \end{pmatrix}$.

For example: $\vec{PQ} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ is perpendicular to $\vec{RS} = \begin{pmatrix} -4 \\ -6 \end{pmatrix}$ when $\begin{pmatrix} -4 \\ -6 \end{pmatrix} = k\begin{pmatrix} -2 \\ -3 \end{pmatrix}$. That is $\begin{pmatrix} -4 \\ -6 \end{pmatrix} = \begin{pmatrix} -2k \\ -3k \end{pmatrix}$.

This implies that $-2k = -4 \Rightarrow k = 2$ or $-6 = -3k \Rightarrow k = 2$ (*i.e.*, the value of k must be the same).



This type of relationship may be represented in the above figure, where vector $\vec{AB}$ is perpendicular to vector $\vec{PQ}$.

Example 6. *Which of the vector is perpendicular to the vector* $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$?

(*a*) $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ (*b*) $\begin{pmatrix} -1 \\ -3 \end{pmatrix}$ (*c*) $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ (*d*) $\begin{pmatrix} -3 \\ -1 \end{pmatrix}$

Solution. The vectors perpendicular to $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ are

$\begin{pmatrix} -3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ or their scalar multiples

$\begin{pmatrix} -3k \\ -k \end{pmatrix}$ and $\begin{pmatrix} 3k \\ k \end{pmatrix}$ where k is a positive number. Therefore the correct option is (*d*).

EXERCISE 9.2

1. (*a*) If $\vec{AB} = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$, find $\vec{BA}$ (*b*) If $\vec{AB} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$, find $\vec{BA}$.

2. (*a*) If $\vec{PQ} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$, find $\vec{QP}$ (*b*) If $\vec{DC} = \begin{pmatrix} -3 \\ 7 \end{pmatrix}$, find $-\vec{CD}$.

3. If P(–1, 3) and Q(2, 5), find $\vec{PQ}$.

4. If P(–6, 8) and Q(–10, 6), find $\vec{PQ}$.

5. Given that $\mathbf{s} = \begin{pmatrix} x-4 \\ 3 \end{pmatrix}$ and $\mathbf{t} = \begin{pmatrix} 5 \\ 3-y \end{pmatrix}$, find x and y if **s** = **t.**

6. If $\mathbf{a} = \begin{pmatrix} 5-x \\ y+3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ -6 \end{pmatrix}$, find x and y when **a** = **b**.

7. Which of the following is parallel to $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$?

(*a*) $\begin{pmatrix} 12 \\ 9 \end{pmatrix}$ (*b*) $\begin{pmatrix} -8 \\ 6 \end{pmatrix}$ (*c*) $\begin{pmatrix} 8 \\ 6 \end{pmatrix}$ (*d*) $\begin{pmatrix} -8 \\ -6 \end{pmatrix}$

8. If $\vec{PQ} = \begin{pmatrix} 24 \\ x \end{pmatrix}$ and $\vec{CD} = \begin{pmatrix} 16 \\ 20 \end{pmatrix}$, find x if $\vec{PQ}$ is parallel to $\vec{CD}$.

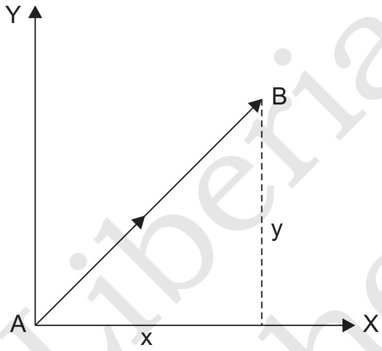
9. Which of the following vectors is perpendicular to the vector $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$?

(*a*) $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ (*b*) $\begin{pmatrix} 12 \\ 16 \end{pmatrix}$ (*c*) $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ (*d*) $\begin{pmatrix} 8 \\ -6 \end{pmatrix}$

10. The vector $\begin{pmatrix} 9 \\ x \end{pmatrix}$ is perpendicular to the vector $\begin{pmatrix} 6 \\ 3 \end{pmatrix}$. Find the value of x.

9.3 MAGNITUDE *AND* DIRECTION OF A VECTOR

Magnitude or length of a vector



If $\vec{AB} = \begin{pmatrix} x \\ y \end{pmatrix}$, then the magnitude of $\vec{AB}$ denoted by $|\vec{AB}|$ or AB is given by:

$$|\vec{AB}| = AB = \sqrt{x^2 + y^2}$$

This is found by using Pythagoras theorem.

Exmaple 7. *If* $\vec{CD} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ *and* $\vec{AB} = \begin{pmatrix} 5 \\ 12 \end{pmatrix}$, *find* $|\vec{CD}|$ *and* $|\vec{AB}|$.

Solution.
$$|\vec{CD}| = \sqrt{(-3)^2 + (4)^2}$$
$$= \sqrt{9+16} = \sqrt{25} = 5 \text{ units}$$
$$|\vec{AB}| = \sqrt{(5)^2 + (12)^2}$$
$$= \sqrt{(25+144)} = \sqrt{169} = 13 \text{ units.}$$

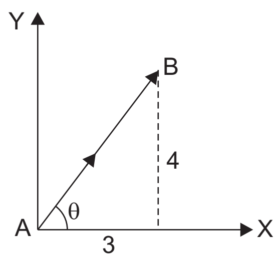
Example 8. *Find the length of the vector* $\vec{AB} = \begin{pmatrix} -5 \\ 10 \end{pmatrix}$.

Solution. Length of vector AB = $|\vec{AB}|$

$$\therefore \quad |\vec{AB}| = \sqrt{(-5)^2 + (10)^2}$$
$$= \sqrt{25+100} = \sqrt{125} = 5\sqrt{5} \text{ units.}$$

Direction of a Vector

The direction is measured from the *north* in the *clockwise direction.*



To find the direction of a vector, $\vec{AB} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, first represent the vector on a rough diagram, and find the acute angle it makes with the *x*-axis using the relation for tangent of an angle in a right-angled triangle. Note that both the *x* and *y* components of the vector are positive meaning the vector is in the first quadrant.

From the diagram,

$\tan\theta = \frac{4}{3} = 1.33 \Rightarrow \theta = 53°$. [Using table of tangents of angles]

$\therefore$ The direction of $\vec{AB}$ measured from the north

$= 90° - 53° = 37°$.

Example 9. *Find the magnitude and directions (bearings) of the following column vectors.*

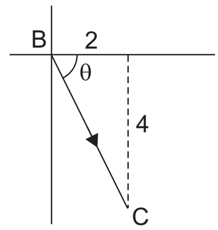
(*i*) $\vec{BC} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$ (*ii*) $\vec{PQ} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ (*iii*) $\vec{QR} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

Note: Bearings give directions in terms of an angle.

Solution. (*i*) $|\vec{BC}| = \sqrt{(2)^2 + (-4)^2}$

$= \sqrt{4 + 16} = \sqrt{20} = 4.47$ units

For the direction of $\vec{BC}$, first represent the vector on a rough diagram, and find the acute angle it makes with the *x*-axis using the relation for tangent of an angle in a right angled triangle.



From the diagram, $\tan\theta = \frac{4}{2} = 2$

$\Rightarrow \theta = 63.4°$ [Using table of tangents of angles]

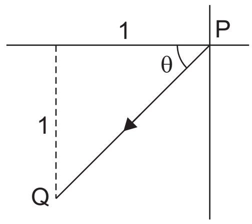
$\therefore$ The direction of $\vec{BC}$ measured from the north

$= 90° + 63.4° = 153.4°$.

(*ii*) $|\vec{PQ}| = \sqrt{(-1)^2 + (-1)^2}$

$= \sqrt{(1+1)} = \sqrt{2} = 1.41$ units

For the direction of $\vec{PQ}$, first represent the vector on a rough diagram, and find the acute angle it makes with the *x*-axis using the relation for tangent of an angle in a right-angled triangle.



From the diagram, $\tan\theta = \frac{1}{1} = 1$

$\Rightarrow \quad \theta = 45°$ [Using table of tangents of angles]

$\therefore$ The direction of $\vec{PQ}$ measured from the north

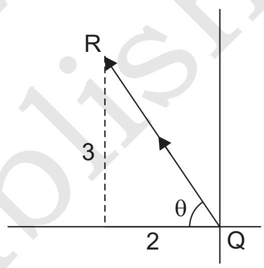
$= 270° - 45° = 225°$

(*iii*) $|\vec{QR}| = \sqrt{(-2)^2 + (3)^2}$

$= \sqrt{(4+9)} = \sqrt{13} = 3.6$ units.

For the direction of $\vec{QR}$, first represent the vector on a rough diagram, and find the acute angle it makes with the *x*-axis.

Using the relation for tangent of an angle in a right-angled triangle.



From the diagram, $\tan\theta = \frac{3}{2} = 1.5$

$\Rightarrow$ $\theta = 56°$ to the nearest degree. [Using table of tangents of angles]

$\therefore$ The direction of $\vec{QR}$ measured from the north

$= 270° + 56° = 326°$.

EXERCISE 9.3

1. Find the magnitude of the following vectors:

(*a*) $\vec{AB} = \begin{pmatrix} 12 \\ 13 \end{pmatrix}$ (*b*) $\vec{CD} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$ (*c*) $\vec{EF} = \begin{pmatrix} -5 \\ 12 \end{pmatrix}$ (*d*) $\vec{PQ} = \begin{pmatrix} 9 \\ 12 \end{pmatrix}$

2. Given the vector $\vec{PQ} = \begin{pmatrix} -7 \\ 12 \end{pmatrix}$, calculate the:

(*a*) length $|\vec{PQ}|$, correct to three significant figures

(*b*) bearing of Q from P, correct to the nearest degree.

3. If P(–1, 2) and Q(*x*, *y*) are point in the O*xy* plane such that $\vec{QP} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$. Find

(*a*) the co-ordinate of Q (*b*) $|\vec{QP}|$

(*c*) the bearing of Q from P.

4. If P(2, 2) and Q(5, 4)

(*a*) Calculate the magnitude of $\overrightarrow{PQ}$.

(*b*) Express $\overrightarrow{PQ}$ in the form (k, θ), where k is the magnitude and θ the bearing.

9.4 ADDITION *AND* SUBTRACTION OF VECTORS

ACTIVITY 1

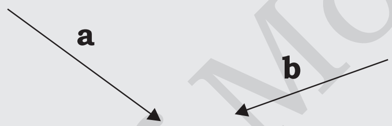
To add **a** and **b**:

Step 1: Draw **a.**

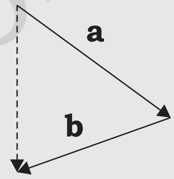
Step 2: At the arrowhead end of **a**, draw **b**.

Step 3: Join the beginning of **a** to the arrowhead end of **b**. This is vector **a + b**.

So, given



we have **a + b**



For example: If $\mathbf{a} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, then

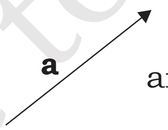
$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$$

[Add the corresponding components]

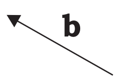
To subtract one vector from another, we simply *add its negative.*

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\,\mathbf{b})$$

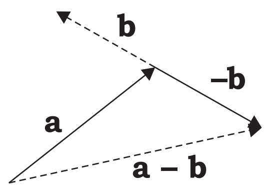
If we have



and



then



For example: If $\mathbf{a} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, then

$$\mathbf{a} - \mathbf{b} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix}$$

[Subtract the corresponding components]

Note: While adding two vectors, the end point of the first vector must be the same as the starting point of the second vector.

Example 10. *If* $\mathbf{r} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ *and* $\mathbf{s} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, *then find:*

(*i*) **r + s** (*ii*) **r – s**

Solution. (*i*) $\mathbf{r} + \mathbf{s} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3+1 \\ 4+2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$

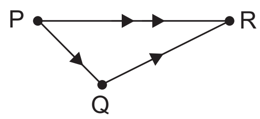
[Add the corresponding components]

(*ii*) $\mathbf{r} - \mathbf{s} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3-1 \\ 4-2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

[Subtract the corresponding components]

Example 11. *If* $\vec{PQ} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ *and* $\vec{RQ} = \begin{pmatrix} -5 \\ -8 \end{pmatrix}$, *find* $\vec{PR}$.

Solution. From the given vectors, $\vec{PR} = \vec{PQ} + \vec{QR}$

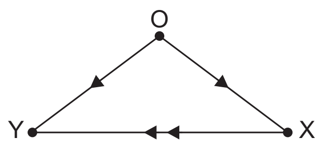


$\vec{PQ}$ has been given but $\vec{QR} = -\vec{RQ} = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$

$\therefore \quad \vec{PR} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} + \begin{pmatrix} 5 \\ 8 \end{pmatrix} = \begin{pmatrix} 4 \\ 11 \end{pmatrix}.$

Example 12. *If* $\vec{OX} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ *and* $\vec{OY} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, *find* $|XY|$.

Solution. From the given vectors, $\vec{XY} = \vec{XO} + \vec{OY}$



$\vec{OY}$ has been given

But $\quad \vec{XO} = -\vec{OX} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}$ [Negative vectors]

$\Rightarrow \quad \vec{XY} = \begin{pmatrix} -4 \\ -3 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$

$\therefore \quad |XY| = \sqrt{(-2)^2 + (-4)^2}$

$= \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$ units.

9.5 SCALAR MULTIPLICATION

If $\vec{AB} = \begin{pmatrix} x \\ y \end{pmatrix}$, then $k\vec{AB} = \begin{pmatrix} kx \\ ky \end{pmatrix}$, where k is a *scalar* or *number* which can be a negative or positive whole number or fraction. When k is positive, it implies the vectors are parallel and in the same direction. When k is negative, it implies the vectors are parallel but in opposite directions. The length of the new vector is $|\,k\,|$ times the length of the original vector.

Note: To find the scalar multiple of a vector, multiply each component of the vector by the scalar.

Example 13. *If* $\vec{AB} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$, *find (i)* $2\vec{AB}$ *(ii)* $-3\vec{AB}$.

Solution. (*i*) $2\vec{AB} = 2\begin{pmatrix} -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 2\times(-1) \\ 2\times 5 \end{pmatrix} = \begin{pmatrix} -2 \\ 10 \end{pmatrix}$

(*ii*) $-3\vec{AB} = -3\begin{pmatrix} -1 \\ 5 \end{pmatrix} = \begin{pmatrix} -3\times(-1) \\ -3\times 5 \end{pmatrix} = \begin{pmatrix} 3 \\ -15 \end{pmatrix}$.

Example 14. *If* $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}$ *and* $\mathbf{c} = \begin{pmatrix} -2 \\ -6 \end{pmatrix}$, *find*

(*i*) $2\mathbf{b}$ (*ii*) $\mathbf{a} - 2\mathbf{b} + \mathbf{c}$ (*iii*) $2(\mathbf{a} + \mathbf{b})$

Solution. (*i*) $2\mathbf{b} = 2\begin{pmatrix} 4 \\ -5 \end{pmatrix} = \begin{pmatrix} 8 \\ -10 \end{pmatrix}$

(*ii*) $\mathbf{a} - 2\mathbf{b} + \mathbf{c} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 4 \\ -5 \end{pmatrix} + \begin{pmatrix} -2 \\ -6 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 8 \\ -10 \end{pmatrix} + \begin{pmatrix} -2 \\ -6 \end{pmatrix}$

$$= \begin{pmatrix} 3 - 8 + (-2) \\ 1 - (-10) + (-6) \end{pmatrix} = \begin{pmatrix} -7 \\ 5 \end{pmatrix}.$$

(*iii*) First find the expression in the bracket i.e. $\mathbf{a} + \mathbf{b}$

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ -5 \end{pmatrix} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$$

$$\therefore \quad 2(\mathbf{a} + \mathbf{b}) = 2\begin{pmatrix} 7 \\ -4 \end{pmatrix} = \begin{pmatrix} 14 \\ -8 \end{pmatrix}.$$

EXERCISE 9.4

1. If $\mathbf{a} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$, find (*a*) $\mathbf{a} + \mathbf{b}$ (*b*) $\mathbf{a} - \mathbf{b}$.

2. Given that $\mathbf{a} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$, find (*a*) $\mathbf{a} + \mathbf{b} - \mathbf{c}$ (*b*) $\mathbf{b} - \mathbf{c} + \mathbf{a}$.

3. If $\overrightarrow{AB} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$, find $\overrightarrow{AC}$.

4. If $\overrightarrow{XY} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$ and $\overrightarrow{ZY} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$, find $\overrightarrow{XZ}$.

5. Given that $\overrightarrow{PQ} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$ and $\overrightarrow{RQ} = \begin{pmatrix} 6 \\ -5 \end{pmatrix}$, find $|\overrightarrow{PR}|$.

6. If $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{s} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, calculate $6(\mathbf{r} + 2\mathbf{s})$.

7. If $\mathbf{p} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, find $2\mathbf{p} - \mathbf{q} + \mathbf{r}$.

8. If $\mathbf{r} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\mathbf{s} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\mathbf{t} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$, find $2\mathbf{r} - \mathbf{s} + \mathbf{t}$.

REVIEW EXERCISE

1. Express the following vectors graphically.

   (*a*) $\overrightarrow{AB}$ = (3 km, 245°) (*b*) $\overrightarrow{BC} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

2. If $\boldsymbol{p} = \begin{pmatrix} 3x+1 \\ 2y+3 \end{pmatrix}$ and $\boldsymbol{q} = \begin{pmatrix} -8 \\ 5 \end{pmatrix}$, find x and y, if $\boldsymbol{p} = \boldsymbol{q}$.

3. If $\overrightarrow{BA} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$, find $\overrightarrow{AB}$.

4. If $\mathbf{p} = \begin{pmatrix} 4x+3 \\ 5+2y \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 11 \\ -5 \end{pmatrix}$, find x and y when $\mathbf{p} = \mathbf{q}$.

5. Which of the following is parallel and opposite to $\begin{pmatrix} 2 \\ -6 \end{pmatrix}$?

   (*a*) $\begin{pmatrix} 4 \\ 12 \end{pmatrix}$ (*b*) $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ (*c*) $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ (*d*) $\begin{pmatrix} -4 \\ -12 \end{pmatrix}$

6. The vector $\begin{pmatrix} 12 \\ x \end{pmatrix}$ is parallel to the vector $\begin{pmatrix} 8 \\ 6 \end{pmatrix}$. Find the value of *x*.

7. Find the magnitude of the following vectors:

(*a*) $\overrightarrow{LM} = \begin{pmatrix} -7 \\ 24 \end{pmatrix}$ (*b*) $\overrightarrow{XY} = \begin{pmatrix} 15 \\ 8 \end{pmatrix}$

8. If $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}$ then, find (*a*) $\mathbf{a} + \mathbf{b}$ (*b*) $\mathbf{a} - \mathbf{b}$.

9. If $\overrightarrow{XY} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$ and $\overrightarrow{ZY} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, find $\overrightarrow{XZ}$.

10. Given that $\mathbf{p} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$, find (*a*) $\mathbf{p} + \mathbf{q}$ (*b*) $\mathbf{q} - \mathbf{p}$.

11. If $\overrightarrow{PQ} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and $\overrightarrow{RQ} = \begin{pmatrix} -1 \\ -4 \end{pmatrix}$, find $\overrightarrow{PR}$.

12. Given that $\overrightarrow{PQ} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\overrightarrow{RQ} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$, find $\overrightarrow{PR}$.

13. If $\mathbf{a} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, then find (*a*) $-3\mathbf{a}$ (*b*) $2\mathbf{a}$

14. If $\mathbf{a} = \begin{pmatrix} 3 \\ -6 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, find $2\mathbf{a} + 3\mathbf{b}$.

MULTIPLE CHOICE QUESTIONS (MCQs)

1. Simplify $\begin{pmatrix} 2 \\ -3 \end{pmatrix} - \begin{pmatrix} -1 \\ 5 \end{pmatrix}$

(a) $\begin{pmatrix} -3 \\ 8 \end{pmatrix}$ (b) $\begin{pmatrix} 3 \\ -8 \end{pmatrix}$ (c) $\begin{pmatrix} -3 \\ -8 \end{pmatrix}$ (d) $\begin{pmatrix} 3 \\ 8 \end{pmatrix}$

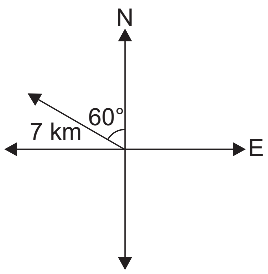
2. If $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{s} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, calculate $6(\mathbf{r} + 2\mathbf{s})$

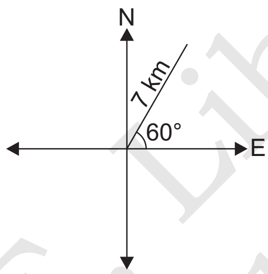
(a) $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ (b) $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ (c) $\begin{pmatrix} 7 \\ 3 \end{pmatrix}$ (d) $\begin{pmatrix} -6 \\ 18 \end{pmatrix}$

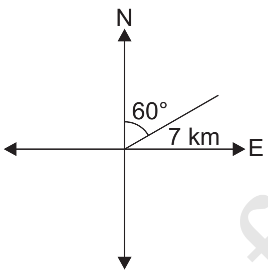
3. Find the length of the vector P = $\begin{pmatrix} -5 \\ 12 \end{pmatrix}$

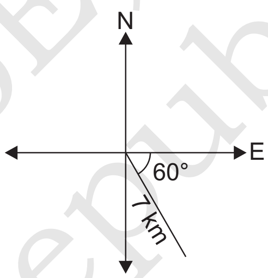
(a) 17 (b) 13 (c) 25 (d) 7

4. A boy walked 7 km on a bearing 60°. Which of the following diagrams shows his direction?

(a)


(b)


(c)


(d)


5. If P(2, 5) and Q(–2, 3) are points in the cartesian plain, find the vector $\overrightarrow{PQ}$.

(a) $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$ (b) $\begin{pmatrix} -4 \\ -2 \end{pmatrix}$ (c) $\begin{pmatrix} -4 \\ 2 \end{pmatrix}$ (d) $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$

6. Find k in the vector equation, $\begin{pmatrix} 3 \\ 4 \end{pmatrix} + k\begin{pmatrix} 3 \\ 4 \end{pmatrix} = -\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

(a) $-\frac{3}{4}$ (b) –3 (c) 2 (d) –2

7. Simplify $\begin{pmatrix} 4 \\ 5 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 6 \end{pmatrix}$

(a) $\begin{pmatrix} 7 \\ 13 \end{pmatrix}$ (b) $\begin{pmatrix} 5 \\ 13 \end{pmatrix}$ (c) $\begin{pmatrix} 2 \\ 13 \end{pmatrix}$ (d) $\begin{pmatrix} 3 \\ 13 \end{pmatrix}$

8. If $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{s} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$. Calculate $2\mathbf{r} - 3\mathbf{s}$.

(a) $\begin{pmatrix} 18 \\ -8 \end{pmatrix}$ (b) $\begin{pmatrix} -5 \\ 12 \end{pmatrix}$ (c) $\begin{pmatrix} 13 \\ -4 \end{pmatrix}$ (d) $\begin{pmatrix} -5 \\ -8 \end{pmatrix}$

9. If $\mathbf{u} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$, find $2\mathbf{u} + 3\mathbf{v}$.

(a) $\begin{pmatrix} 11 \\ 5 \end{pmatrix}$ (b) $\begin{pmatrix} 11 \\ 13 \end{pmatrix}$ (c) $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$ (d) $\begin{pmatrix} 14 \\ 0 \end{pmatrix}$

10. If $\mathbf{u} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$, find $2\mathbf{u} + \mathbf{v}$.

(a) $\begin{pmatrix} 12 \\ 3 \end{pmatrix}$ (b) $\begin{pmatrix} 12 \\ -1 \end{pmatrix}$ (c) $\begin{pmatrix} 12 \\ 0 \end{pmatrix}$ (d) $\begin{pmatrix} 12 \\ 1 \end{pmatrix}$

RECAP AT A GLANCE

- A quantity that has only magnitude is called a *scalar.*
- A quantity that has magnitude as well as a direction is called a *vector.*
- A *vector* is any physical quantity that has *length* (or magnitude) and *direction.*
- A *zero vector* is a vector which *begins* and *ends* at the *same point.*
- A *position vector* is a vector which begins from the *origin* and ends at a point.
- Negative vector is the *opposite* or *inverse* of a vector.
- If any two or more vectors have the same components, then they are *equal vectors.*
- Two or more vectors are parallel, if one is a *scalar multiple* of the other(s).
- The vectors perpendicular to $\vec{AB} = \begin{pmatrix} a \\ b \end{pmatrix}$ are $\begin{pmatrix} b \\ -a \end{pmatrix}$ and $\begin{pmatrix} -b \\ a \end{pmatrix}$ or their scalar multiples $\begin{pmatrix} kb \\ -ka \end{pmatrix}$ and $\begin{pmatrix} -kb \\ ka \end{pmatrix}$.
- If $\vec{AB} = \begin{pmatrix} x \\ y \end{pmatrix}$, then the magnitude of $\vec{AB}$ denoted by $|\vec{AB}|$ or AB is given by:

$$|\vec{AB}| = AB = \sqrt{x^2 + y^2}$$

- To find the scalar multiple of a vector, multiply each component of the vector by the scalar.

❑ ❑ ❑

PERIOD-VI

M10CH10

TOPIC 10

Rigid Motion

10.1 RIGID MOTION

A *rigid body* which does not deform under the influence of forces is known as rigid body.

In real life, no object is a rigid body. A bridge, however, does not deform under the influence of a single man, but it deforms under the influence of a truck or trucks. So, the bridge is not a rigid body.

A *rigid motion* is an action of taking an object and moving it to a different location without altering its shape or size.

In rigid motion, the distance between any two points of the rigid body remains constant before and after applying forces on it.

Types of Rigid Motion

The following are the types of a rigid motion:

(*a*) translation (*b*) reflection (*c*) rotation

We shall discuss here translation and reflection of rigid body motions in details in our subsequent sections.

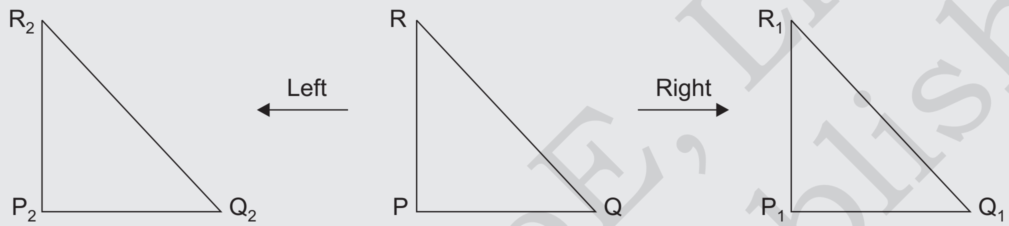
10.2 TRANSLATION

ACTIVITY 1

- To demonstrate as a movement in a straight line, draw a triangle on a piece of cardboard and cut it out.
- Place the triangle on a sheet of paper on a table and draw the outline of the triangle on the paper.
  Label this outline PQR.

- Put the triangular card on the outline again and place a straight edge along the base PQ.
- Slide the triangular card along the straight edge to the right and draw the outline on the sheet of paper to give the image $\Delta P_1Q_1R_1$.
- Again slide it to the left and draw the image $\Delta P_2Q_2R_2$.
- Measure the lengths and angles of $\Delta P_1Q_1R_1$, and $\Delta P_2Q_2R_2$.

What do you observe about the corresponding lengths and angles of the triangles?



This activity shows that the movement of triangle ABC is a translation. When you walk from one place to another, you are making a tranalation. When you push a table away from a place or when you lift a bucket of water, you are making a translation.

Thus, a rigid motion which drags an object in a specified direction and by a specified amount is known as translation.

A translation is always described by a *vector.* The direction and amount of dragging can be represented by a vector known as *translation vector* or *displacement vector*. It is denoted by $\begin{pmatrix} a \\ b \end{pmatrix}$ where a is the horizontal displacement either to right or to left and b is the vertical displacement either upward or downward. Thus, if a point P(x, y) is translated to a point P′ with translation vector $\begin{pmatrix} a \\ b \end{pmatrix}$, then the coordinates of P′ are P′(x + a, y + b). In other words,

$$\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{\text{translation}} \begin{pmatrix} x+a \\ y+b \end{pmatrix}.$$

Remark: The point P′ is called the *image* of P and P is called the *pre-image* of P′ under translation.

A move upward or to the right is indicated by +ve sign and downward or to the left is indicated by –ve sign.

For example: A point P is translated (moved) by the vector $\begin{pmatrix} -2 \\ -9 \end{pmatrix}$ to give its image P′ (as shown in the figure (*i*)).

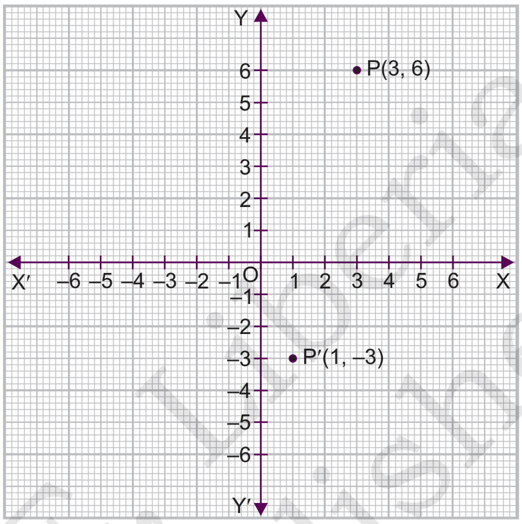


Figure (*i*)

Explanation: Here the coordinates of the point P are P(3, 6) and the translation vector is $\begin{pmatrix} -2 \\ -9 \end{pmatrix}$ therefore, the coordinates of P are P′(3 – 2, 6 – 9) or P′(1, –3)

For example: A line AB has been translated (moved) –2 units to the left (horizontal displacement) and –9 units downward (vertical displacement) to get the line A′B′ (as shown in the figure (*ii*)).

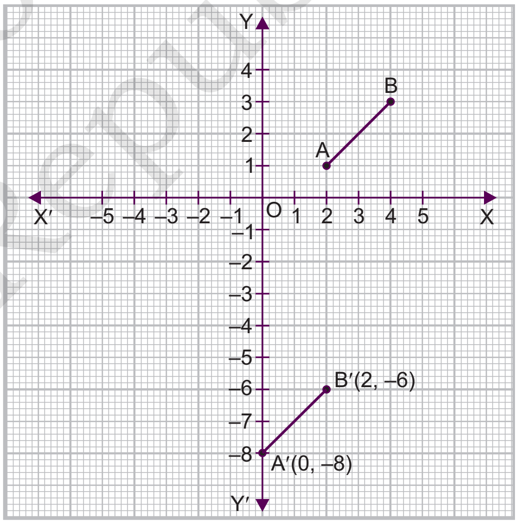


Figure (*ii*)

Explanation: The end points of the line AB are A(2, 1) and B(4, 3). The translation vector is $\begin{pmatrix} -2 \\ -9 \end{pmatrix}$.

Therefore,

A(2, 1) → A′(2 – 2, 1 – 9) = A′(0, –8)

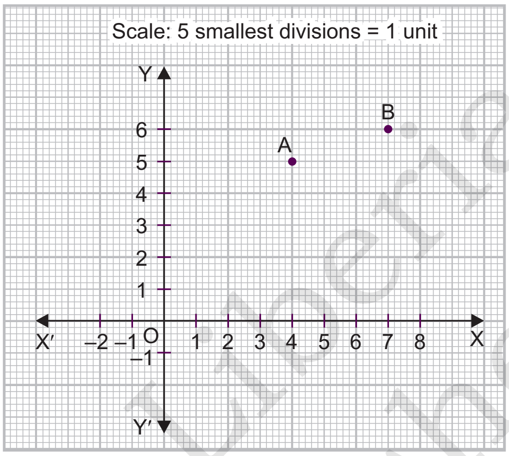
B(4, 3) → B′(4 – 2, 3 – 9) = B(2, –6)

Hence, the translated line A′B′ is shown in the above figure (*ii*).

Finding the Image When the Point and the Translation Vector are Given

Example 1. *Translate the point A(4, 5) to the point (image) B by the translation vector* $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

Solution. Plot the point A(4, 5) on a graph sheet. Move A, 3 units horizontally and 1 unit vertically to the position B. The coordinates of this new position are (7, 6) as shown in the figure.



Alternate method

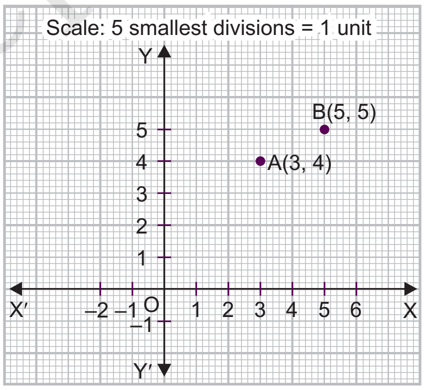
Under a translation by the vector $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 4 \\ 5 \end{pmatrix} \to \begin{pmatrix} 4+3 \\ 5+1 \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \end{pmatrix} \quad \text{or} \quad (7, 6)$$

Therefore, the image of $A(4, 5)$ under the translation vector $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ is B(7, 6) (see the figure).

Finding the Point When Image and Translation Vector are Given

Example 2. *If B (5, 5) is the image of a point A under the translation by the vector* $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, *then find the point A.*



Solution. Suppose the coordinates of A are (x, y).

Under a translation by the vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$,

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x+2 \\ y+1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$

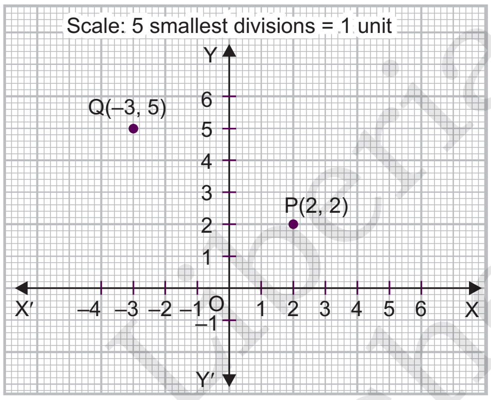
Therefore, from the equality of vectors,

$$x + 2 = 5 \quad \text{and} \quad y + 1 = 5$$

$$\Rightarrow x = 3 \quad \Rightarrow y = 4$$

Hence, the point A has coordinates (3, 4) (see the figure).

Finding the Translation Vector



Example 3. *If Q (–3, 5) is the image of a point P(2, 2) under a translation by a vector as shown on the number plane, find the translation vector (see the figure).*

Solution. Let the translation vector be

$$\begin{pmatrix} a \\ b \end{pmatrix}, \text{ then } \quad \begin{pmatrix} 2 \\ 2 \end{pmatrix} \rightarrow \begin{pmatrix} 2+a \\ 2+b \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$$

Therefore, from the equality of vectors,

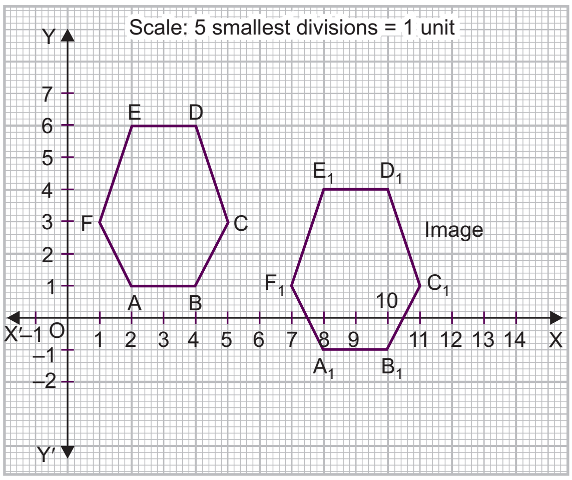
$$2 + a = -3 \quad \text{and} \quad 2 + b = 5$$

$$\Rightarrow \quad a = -5 \quad \Rightarrow \quad b = 3$$

Hence, the translation vector is $\begin{pmatrix} -5 \\ 3 \end{pmatrix}$.

Translation of Plane Figures

Example 4. *Draw a hexagon ABCDEF having vertices A(2, 1), B(4, 1), C(5, 3), D(4, 6), E(2, 6) and F(1, 3). Also draw its image under the translation vector* $\begin{pmatrix} 6 \\ -2 \end{pmatrix}$.



Solution. The vertex A(2, 1) of ABCDEF is translated into vertex A_1 of $A_1B_1C_1D_1E_1F_1$ using the translation vector $\begin{pmatrix} 6 \\ -2 \end{pmatrix}$ as:

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 2+6 \\ 1-2 \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \end{pmatrix}$$

Similarly, the other vertices can be calculated. These are $B_1(10, -1)$; $C_1(11, 1)$; $D_1(10, 4)$, $E_1(8, 4)$; $F_1(7, 1)$.

The hexagon ABCDEF and its image $A_1B_1C_1D_1E_1F_1$ are shown in the figure.

EXERCISE 10.1

1. Find the image A′ if A(3, 4) is translated by the vector $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$.
2. Find the image of (*a*) (1, 2) and (*b*) (– 2, – 4) under the translation by the vector $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$.
3. P′(4, 6) is the image of a point P under the translation by the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Find the point P.
4. P′(–2, 4) is the image of a point P(1, 2) under a translation by a vector. Find the translation vector.
5. Under the translation, the image of the point (5, 4) is (7, 1). What is the image of the point (1, – 4) under the same translation.
6. Q′ is the image of Q(2, 1) under a translation which maps P(3, 4) onto P′(7, 6). Find the coordinates of Q′.
7. P′(8, –2) is the image of the point P(5, 2) by the translation vector v. Find
   (*a*) the vector v
   (*b*) the coordinates of the point Q whch maps onto the point Q′(5, –2) under v
   (*c*) $\overrightarrow{PP'}$

10.3 REFLECTION

A reflection is the image you see when you look in a mirror. The mirror forms the line of symmetry (Discussed further in section 10.4) between the object and the image. Reflection conserves angles, lengths and area but reverses the figure. To define a reflection you need to know the position of the line which the figure is to be reflected.

When an object is reflected in a line, the *image* point is at the opposite side of the line and the perpendicular distance from the point to the line is equal to the perpendicular distance from the image point to the line. The line is called the *mirror line* or line of reflection.

i.e., object distance from mirror line = image distance from mirror line.

In the figure, OABC has been reflected in the y-axis to give $O_1A_1B_1C_1$.

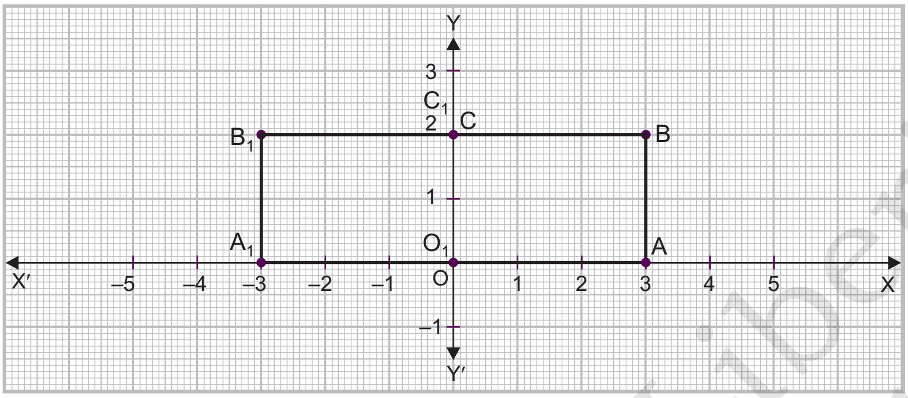


Figure (i)

Also in figure (*ii*), OXYZ has been reflected in the line $y = -x$ to give $O_1X_1Y_1Z_1$

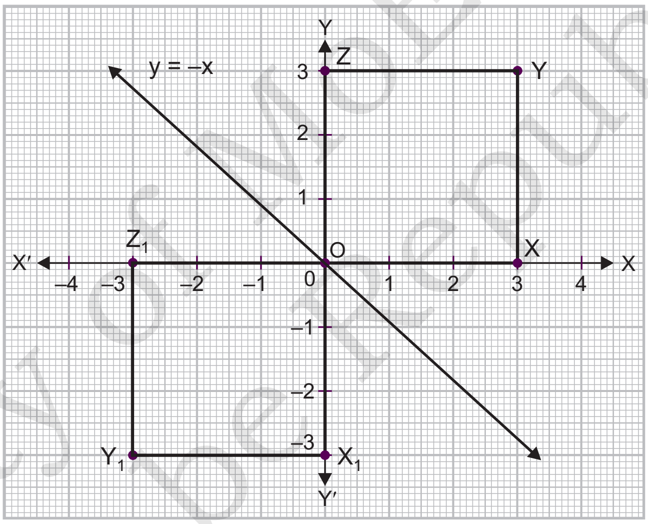


Figure (ii)

Example 5. *What are the properties of objects under reflection with respect to its similarity, congruence and orientation?*

Solution. When an object is reflected, there is no change in the lengths and angles; *i.e.,* the lengths and angles of the object and the corresponding lengths and angles of the image are same.

In other words, *the object and its image are similar as well as congruent.*

There is a change in one aspect between the object and the image, *i.e.,* the left-right changes in the orientation (the shape is same, but the other way round).

We shall consider reflection in the following mirror lines.

(*i*) **Reflection in the *x*-axis (*i.e.* *y* = 0)**

If the point (x, y) is reflected in the x-axis or the line $y = 0$, the image point is $(x, -y)$.

The mapping is: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ -y \end{pmatrix}$ or $(x, y) \to (x, -y)$.

(*ii*) **Reflection in the *y*-axis (i.e. *x* = 0)**

If the point (x, y) is reflected in the y-axis or the line $x = 0$, the image points is $(-x, y)$. The mapping is:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -x \\ y \end{pmatrix} \text{ or } (x, y) \to (-x, y)$$

The general rule is to negate the x-coordinate and maintain the y-coordinate.

For example: Under reflection in the y-axis, $(3, 4) \to (-3, 4)$ and $(-1, -2) \to (1, -2)$.

(*iii*) **Reflection in the line *x* = *k* or *x* – *k* = 0**

Using the same procedure as describe above we can obtain a rule for reflection in the line $x = k$. If the point (x, y) is reflected in the line $x = k$ or the line $x - k = 0$, the image point is $(2k - x, y)$.

The mapping is: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2k - x \\ y \end{pmatrix}$ or $(x, y) \to (2k - x, y)$

For example, under reflection in the line $x = 1$ or $x - 1 = 0$, the value of $k = 1$. Therefore, the mapping becomes $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2 - x \\ y \end{pmatrix}$. The image of the point (3, 4) under the reflection in the line $x = 1$ is:

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} \to \begin{pmatrix} 2 - 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \text{ i.e. } (-1, 4).$$

Example 6. *Find the images of the points A(3, 4) when reflected in the line x – 2 = 0.*

Solution. Note that $x - 2 = 0 \quad \Rightarrow \quad x = 2$

$\therefore \quad k = 2$ and the mapping becomes $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 4 - x \\ y \end{pmatrix}$

$$\begin{pmatrix}3\\4\end{pmatrix} \to \begin{pmatrix}4-3\\4\end{pmatrix} = \begin{pmatrix}1\\4\end{pmatrix}$$

∴ The image of A(3, 4) is (1, 4).

(*iv*) **Reflection in the line *y* = *k* or *y* – *k* = 0**

If the point (x, y) is reflected in the line $y = k$ or the line $y - k = 0$, then the image point is $(x, 2k - y)$. The mapping is:

$$\begin{pmatrix}x\\y\end{pmatrix} \to \begin{pmatrix}x\\2k-y\end{pmatrix} \text{ or } (x, y) \to (x, 2k - y) \text{ where } k \text{ is an integer.}$$

Example 7. *Find the images of the points A(3, 4) when reflected in the line $y + 1 = 0$.*

Solution. Note that $y + 1 = 0 \Rightarrow y = -1$

∴ $k = -1$ and the mapping becomes

$$\begin{pmatrix}x\\y\end{pmatrix} \to \begin{pmatrix}x\\-2-y\end{pmatrix}$$

i.e.,

$$\begin{pmatrix}3\\4\end{pmatrix} \to \begin{pmatrix}3\\-2-4\end{pmatrix} = \begin{pmatrix}3\\-6\end{pmatrix}$$

∴ The image of A(3, 4) is (3, – 6).

(*v*) **Reflection in the line *y* = *kx* or *y* – *kx* = 0**

If the point (x, y) is reflected in the line $y = kx$ or the line $y - kx = 0$, then the image point is $\left(\frac{1}{k}y, kx\right)$. The mapping is:

$$\begin{pmatrix}x\\y\end{pmatrix} \to \begin{pmatrix}\frac{1}{k}y\\kx\end{pmatrix} \text{ or } (x, y) \to \left(\frac{1}{k}y, kx\right)$$

where k is an integer.

Note: $y = kx \Rightarrow x = \frac{1}{k}y$. Therefore the x-coordinate of the image becomes $\frac{1}{k}y$ and the y-coordinate also becomes $2k$.

(*vi*) **Reflection in line *y* = *x***

When $k = 1$, we have *reflection in line $y = x$*, which is given by the mapping

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} y \\ x \end{pmatrix} \text{ or } (x, y) \to (y, x).$$

For example, (3, 4) → (4, 3) and (–1, –2) → (–2, –1).

(*vii*) **Reflection in the line $y = -x$**

Again, when $k = -1$, we have *reflection in the line $y = -x$,* which is given by the mapping:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -y \\ -x \end{pmatrix} \text{ or } (x, y) \to (-y, -x)$$

For example, (3, 4) → (–4, –3) and (–1, –2) → (2, 1).

Example 8. *Find the images of the points A(3, 4) when reflected in the line $y = 2x$.*

Solution. $k = 2$, therefore the mapping:

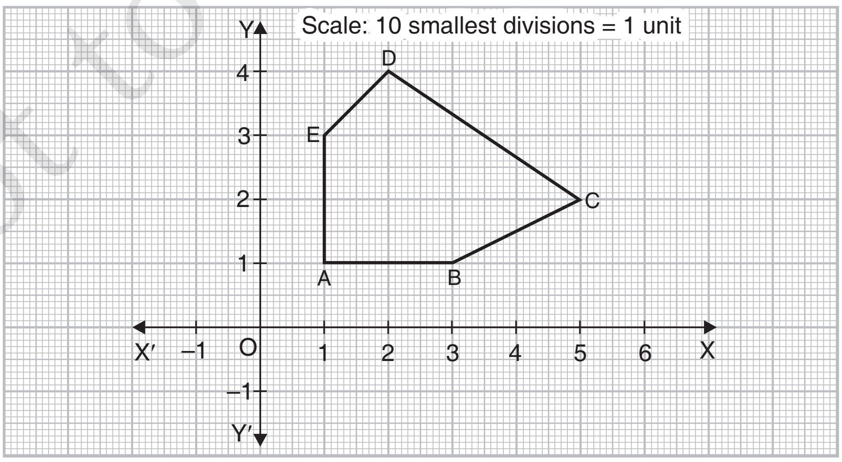
$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \frac{1}{k}y \\ kx \end{pmatrix} \text{ becomes } \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \frac{1}{2}y \\ 2x \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} \to \begin{pmatrix} \frac{1}{2}(4) \\ 2(3) \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix} \text{ or } (2, 6).$$

Reflection of Plane Figures

The reflection of a plane figure in a given mirror line can be done in the Cartesian plane by reflecting the *vertices* of the plane figure in the given mirror line.

Example 9. *Draw and state coordinates of the image of the shape ABCDE in reflection in (i) X-axis (ii) Y-axis in the coordinates plane.*



Solution. (*i*) The reflection of the point (x, y) across the X-axis is the point $(x, -y)$, *i.e.*, $P(x, y) \to P'(x, -y)$

$\therefore$ $A(1, 1) \to A_1(1, -1)$; $B(3, 1) \to B_1(3, -1)$;
$C(5, 2) \to C_1(5, -2)$; $D(2, 4) \to D_1(2, -4)$;
$E(1, 3) \to E_1(1, -3)$

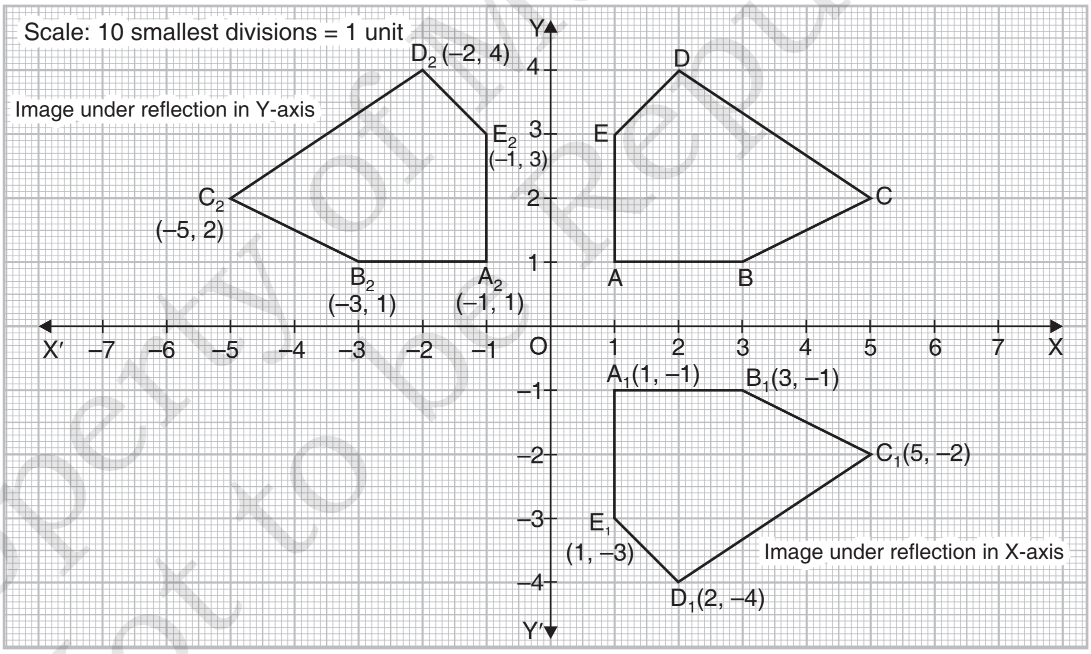
The image of the shape ABCDE in X-axis is $A_1B_1C_1D_1E_1$.

(*ii*) The reflection of the point (x, y) across the Y-axis is the point $(-x, y)$, *i.e.*, $P(x, y) \to P'(-x, y)$.

$\therefore$ $A(1, 1) \to A_2(-1, 1)$; $B(3, 1) \to B_2(-3, 1)$;
$C(5, 2) \to C_2(-5, 2)$; $D(2, 4) \to D_2(-2, 4)$;
$E(1, 3) \to E_2(-1, 3)$

The image of the shap ABCDE in Y-axis is $A_2B_2C_2D_2E_2$.

The following figure illustrates the images of the shape ABCDE in both the axes:



EXERCISE 10.2

1. Find the image of the points

(*a*) B(1, –5) (*b*) P(–1, 2)

when reflected in the line $x - 2 = 0$.

2. Find the image of the points
(*a*) B(1, –5) (*b*) P(–1, 2)
when reflected in the line $y + 1 = 0$.

3. Find the image of the points
(*a*) B(1, –6) (*b*) P(–1, –2)
when reflected in the line $y = 2x$.

4. The coordinates of a rectangle PQRS are P(–4, 1), Q(–1, 1), R(–1, 5), S(–4, 5). Draw and state coordinates of its image in Y-axis.

5. The coordinates of a triangle ABC are A(–3, 4), B(–4, 1) and (–1, 2). Draw and state coordinates of its image in X-axis.

6. Draw and state coordinates of the image of triangle ABC having coordinates A(–1, 4), B(–2, 2), C(4, 3) in the X-axis in the coordinates plane.

7. The coordinates of the image of a shape under reflection in Y-axis are:
$P_1(-2, 3)$, $Q_1(-5, 0)$, $R_1(-3, -2)$.
Draw the original shape in the coordinates plane.

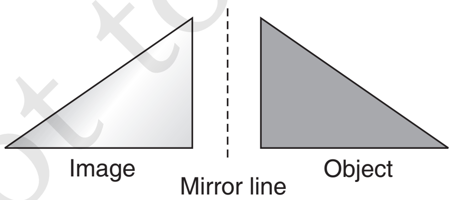
8. The coordinates of the image of a shape under reflection in X-axis are:
$A_1(1, -3)$, $B_1(4, -3)$, $C_1(1, -5)$, $D_1(4, -5)$.
Draw the original shape in the coordinates plane.

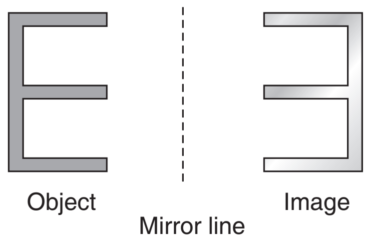
10.4 SYMMETRY

Line of symmetry and mirror reflection are naturally related and linked to each other.

When an object is reflected in a mirror line, the object and its image form a symmetrical shape, with the mirror line as the axis of symmetry.

Look at these figures and their mirror images.





When a mirror is used to reflect an image, the mirror serves as the line of symmetry. Each point on the image will be directly opposite the object and at the same distance on the other side of the mirror line. If a fold were to be made on the mirror line, the object would fall exactly on to the reflected image.

Remember that a reflection is a flip.

Note: Under reflection, an object and its image are congruent but the left-right changes in the orientation.

Designs (or objects) with Reflectional (or fold) Symmetries

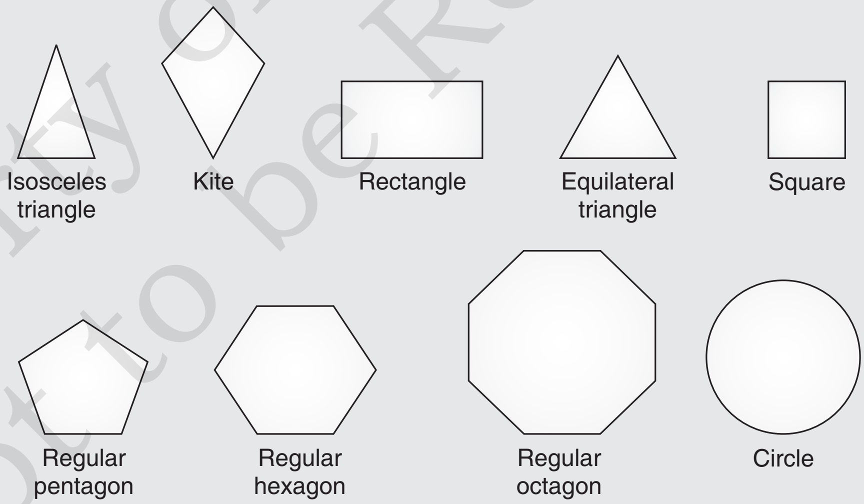
Symmetry is an important geometrical concept, commonly exhibited in nature and is used almost in every field of activity. Artists, professionals, designers of clothing or jewellery car manufacturers, architects and many others make use of the idea of symmetry. The beehives, the flowers, the tree-leaves, religious symbols, rugs, and handkerchiefs—everywhere you find symmetrical designs.

ACTIVITY 2

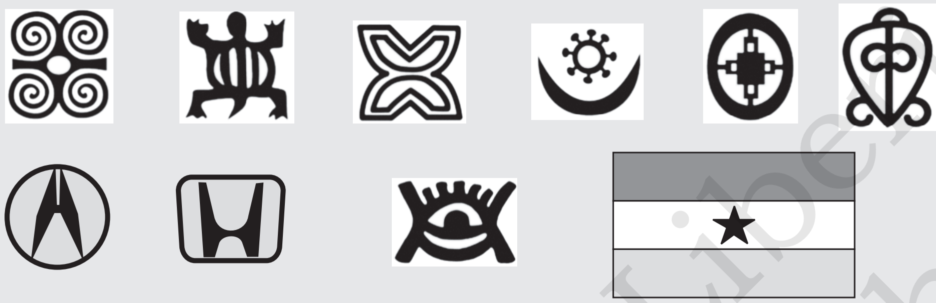
Designs or Objects Used in Everyday Life that have Reflectional (or Fold) Symmetries

Pupils can work individually or in groups to give examples:

- A person's face is one example of symmetry in the real World.
- Many plane shapes have reflectional (fold) symmetries. Here, are a few:



- The designs in adinkra symbols, logos etc. have reflectional symmetry. Here, are a few:

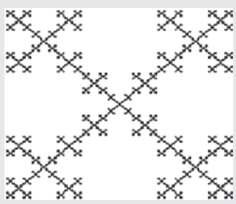


- The designs on some playing cards and artifacts have reflectional (or fold) symmetries as shown below:

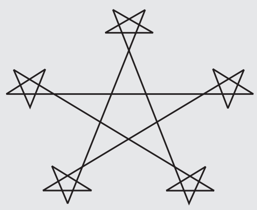
- The manufacturers of some items we use daily have the concept of reflectional (or fold) symmetries. Here, are a few:

- Observe the following beautiful figures. These are symmetrical patterns known as fractals (If you have access to a computer, browse through the topic "Fractals" and find more such beauties):

Barnsley ferm



Box fractal



Star fractal

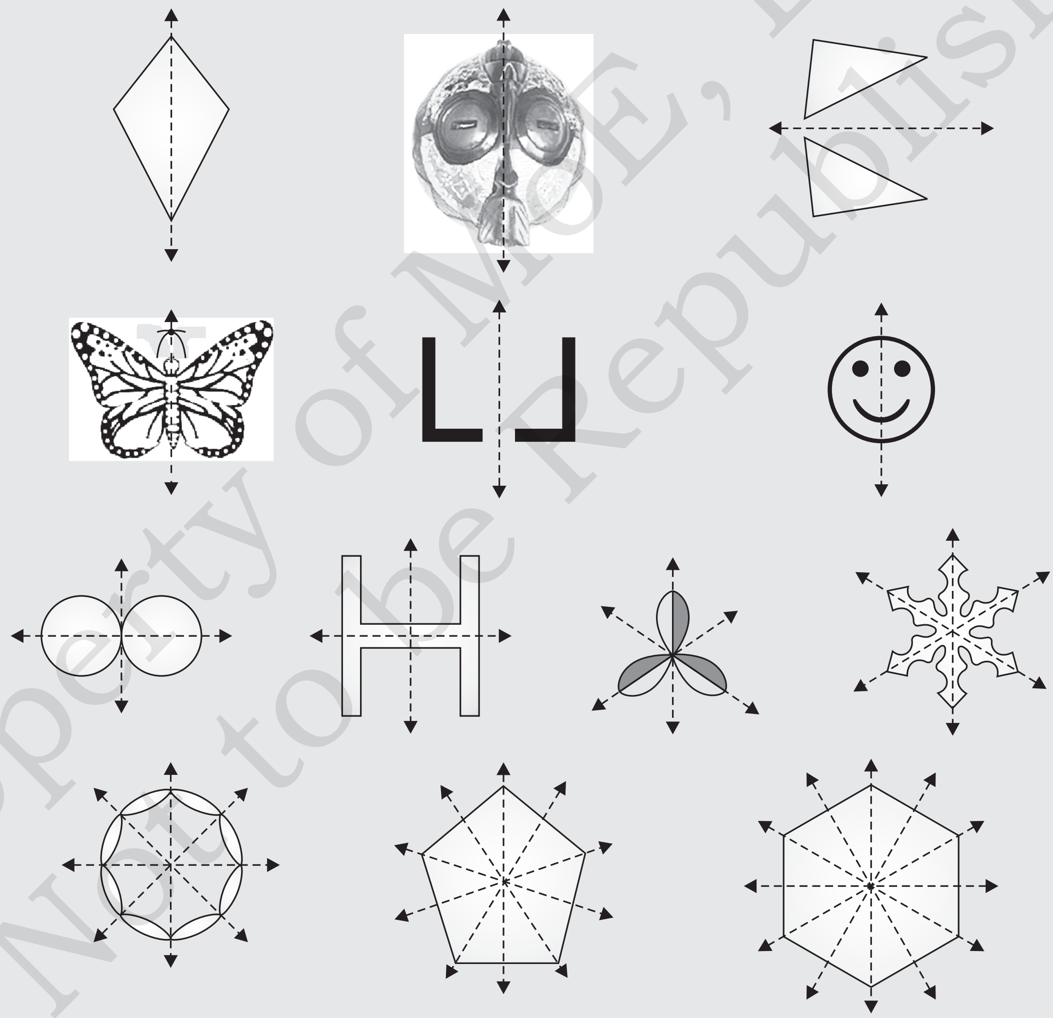
Example 10. *Give some examples of designs or objects in everyday life that have reflectional (or fold) symmetries.*

Solution. Adinkara symbols, logos, light signals at crossings, paper-cut designs etc. are the examples of designs that have reflectional symmetries.

ACTIVITY 3

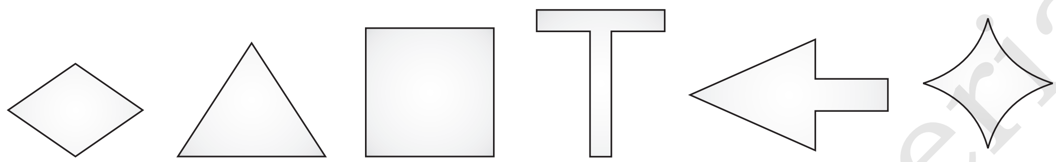
Identifying the Line(s) of Reflection (or Fold) in Objects or Designs

Look at the following objects or designs in the figure.



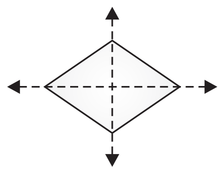
Note: All the dotted lines are the mirror line(s).

Example 11. *Draw and describe the line(s) of symmetry of the following geometric shapes:*

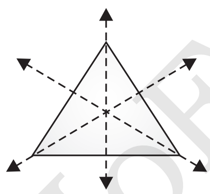


(a) (b) (c) (d) (e) (f)

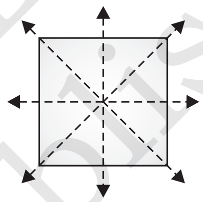
Solution. The line(s) of symmetry of the given geometrical shapes are drawn and described as shown below:



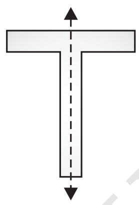
(a) Two lines of symmetry



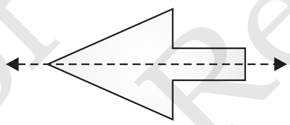
(b) Three lines of symmetry



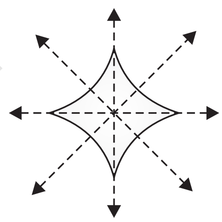
(c) Four lines of symmetry



(d) One line of symmetry

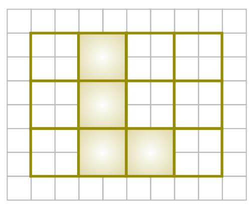


(e) One line of symmetry

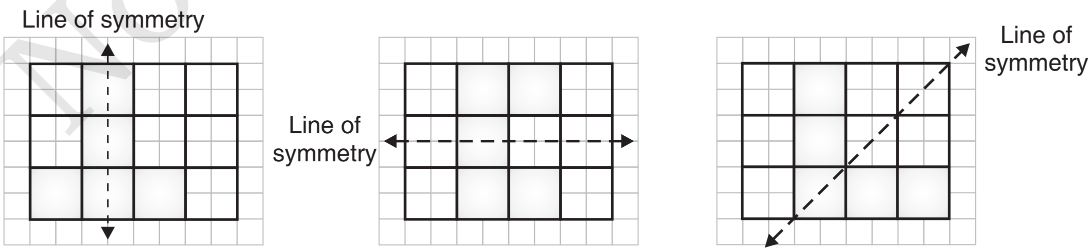


(f) Four lines of symmetry

Example 12. *How many different ways can one more square be shaded in this shape to have a line of symmetry?*



Solution. In the given shape, we can shade one more square in the following different ways to have a line of symmetry:

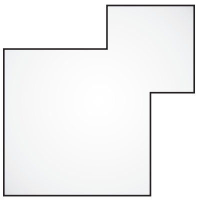


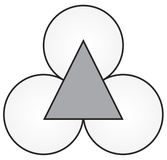
EXERCISE 10.3

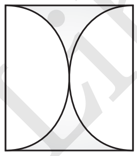
1. Which of the following does not have reflection symmetry.

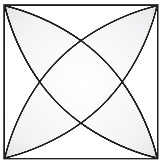
(*a*) T (*b*) V (*c*) A (*d*) R

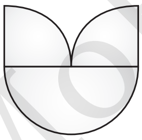
2. Draw and describe the line(s) of symmetry of the following geometric shapes:

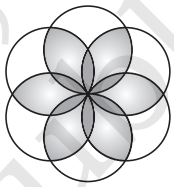

(a)


(b)


(c)


(d)


(e)


(f)

3. Identify which of the following designs in everyday life have reflectional symmetries:


(a)


(b)


(c)

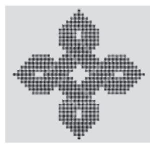

(d)


(e)

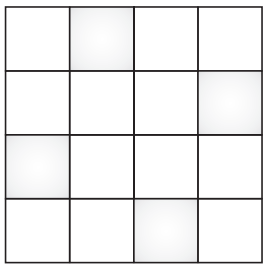

(f)


(g)

(h)


(i)

4. How many different way(s) can four more squares be shaded in this shape to have both the diagonals as line of symmetry? Show the shaded shape also.



REVIEW EXERCISE

1. Given the translation vector $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$ and the coordinates of the image $P_1Q_1R_1S_1T_1$ of shape PQRST as $P_1(6, 1)$, $Q_1(9, 1)$, $R_1(10, 3)$, $S_1(6, 5)$ and $T_1(4, 4)$. Draw the original shape PQRST in the coordinates plane.

2. (*a*) Using a scale of 2 cm to 1 unit of each axis draw on a graph sheet two perpendicular axes OX and OY.

(*b*) On this graph, mark the *x*-axis from – 5 to 5 and the *y*-axis from – 5 to 5.

(*c*) Plot the point A(–1, –1), B(3, 4) and C(2, –1). Join the points to form a triangle.

(*d*) Draw the image of the triangle ABC under the translation by the vector $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$ such that $A \to A_1$, $B \to B_1$ and $C \to C_1$.

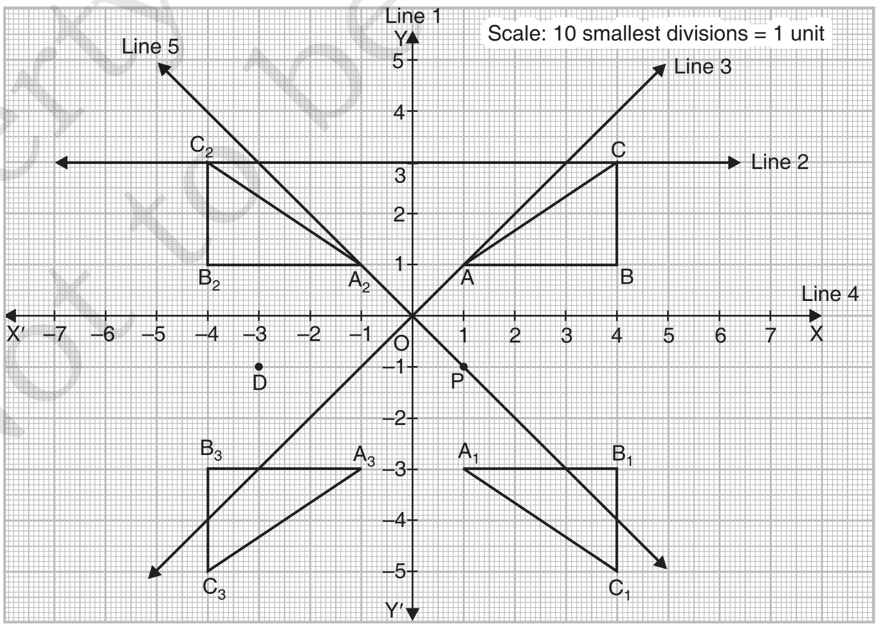
3. The coordinates of the image of a shape under reflection in Y-axis are: $A_1(5, -1)$, $B_1(2, -1)$, $C_1(1, 2)$, $D_1(3, 4)$, $E_1(5, 4)$, $F_1(6, 1)$. Draw the original shape in the coordinates plane.

4. The vertices of a triangle are A(1, 2), B(1, 5) and C(3, 2). Find the vertices A′, B′ and C′ after reflection about

(*a*) $y = x$ (*b*) $y = -x$

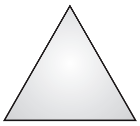
Also, draw the triangles ABC and A′B′C′ in each case.

5. Label the lines shown in the following figure:

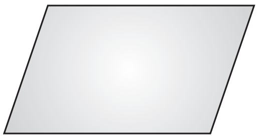


6. State the object coordinates and their corresponding image coordinates in the reflection figure given in question 5.

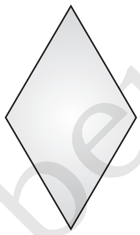
7. Draw and describe the mirror line of the following geometrical shapes:



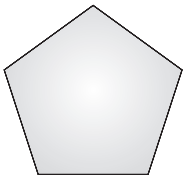
(*a*) Equilateral triangle



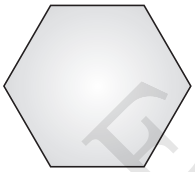
(*b*) Parallelogram



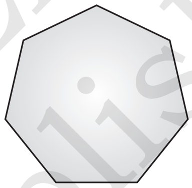
(*c*) Rhombus



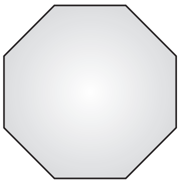
(*d*) Regular pentagon



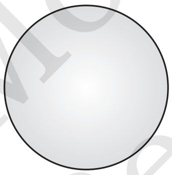
(*e*) Regular hexagon



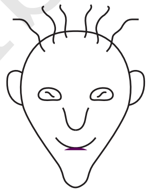
(*f*) Regular heptagon



(*g*) Regular octagon



(*h*) Circle



(*i*) Human face

MULTIPLE CHOICE QUESTIONS (MCQs)

1. Which of these has the least number of lines of symmetry?
(a) An equilateral triangle (b) A rectangle
(c) A square (d) A circle

2. How many lnes of symmetry does a rectangle have?
(a) 1 (b) 2 (c) 3 (d) 4

3. How many lines of symmetry has a square?
(a) 0 (b) 1 (c) 2 (d) 4

4. How many lines of symmetry has an isosceles triangle?
(a) 1 (b) 2 (c) 3 (d) 4

5. If $(x, y) \to (x, 2y)$, find the image of $\left(2\frac{1}{2}, -\frac{1}{4}\right)$ under the same mapping
(a) $\left(2\frac{1}{2}, -2\right)$ (b) $\left(2\frac{1}{2}, -\frac{1}{2}\right)$ (c) $(2, -2)$ (d) $\left(2, -\frac{1}{4}\right)$

6. Find the image of the point, (6, 3) when translated by the vector $\begin{pmatrix} -4 \\ -1 \end{pmatrix}$

(a) (–2, –2) (b) (2, –2) (c) (–2, 2) (d) (2, 2)

7. Find the image of the point K(3, 5) when it is reflected in the *x*-axis.

(a) (3, 5) (b) (5, 3) (c) (3, –5) (d) (–3, –5)

8. A point (2, 1) is reflected in the *y*-axis. Find its image.

(a) (–1, 2) (b) (1, –2) (c) (–2, 1) (d) (2, –1)

RECAP AT A GLANCE

- A *rigid body* which does not deform under the influence of forces is known as rigid body.
- A *rigid motion* is an action of taking an object and moving it to a different location without altering its shape or size.
- A reflection is the image you see when you look in a mirror.
- Line of symmetry and mirror reflection are naturally related and linked to each other.
- Under reflection, an object and its image are congruent but the left-right changes in the orientation.

❑ ❑ ❑

PERIOD-VI

M10CH11

TOPIC

11 Statistics, Ratio and Rates, and Percentages

A. STATISTICS

11.1 STATISTICS

Statistics is the branch of Mathematics which deals with the collection, presentation and analysis of numerical data and drawing conclusion on the basis of the same data.

In particular, statistics deals with how data is:

(*a*) collected;

(*b*) grouped together in a suitable way for graphical representation;

(*c*) interpreted mathematically

Definition of Terms

Discrete data

Values of discrete data are restricted to only certain, exact distinct numbers (but not measurements).

For example: A number of pupils can be 0, 1, 2, 3, 4... but not – 3, 2½, 4.7, 5.8... . Discrete data are usually found by counting.

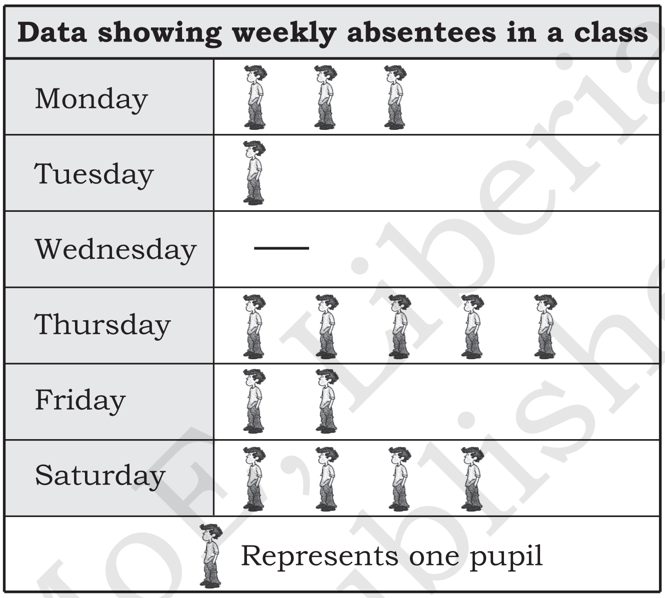
Continuous data

Values of continuous data can be any real number. Continuous data are usually found by measuring (not counting) and rounded to a suitable degree of accuracy.

For example: Weight and height can be measured to the nearest kg or cm respectively.

Collecting Data

The following table shows the data of weekly absentees in a class:



The data about the weekly absentees in a class shown earlier can tell us many things, but it cannot tell us the date which had the maximum number of absentees during a year. To find that, we need to collect the data according to the dates regarding the maximum number of absentees in each of the months during a year.

This shows that a given collection of data may not give us a specific information related to that data. For this we need to collect data keeping in mind that specific information.

Thus, *before collecting data, we need to know what we would use it for.*

ACTIVITY 1

Carrying out simple survey to collect the marks scored in an exercise out of 10

A survey was carried out in a class of 40 pupils. The collected marks are:

8	1	3	7	6	5	5	4	4	2
4	9	5	3	7	1	6	5	2	7
7	3	8	4	2	8	9	5	8	6
7	4	5	6	9	6	4	4	6	6

Note: There are two types of data—*primary data* and *secondary data.*

Primary data is the data collected directly from the source.

For example: If a person wants information about the monthly income, expenditure, number of family members, number of school going children etc. of the employees of a firm.

Secondary data is the data collected from secondary sources such as the Internet, TV, libraries, newspapers, etc.

11.2 FREQUENCY TABLES *AND* HISTOGRAMS

(*a*) Making Frequency Tables

In this section, we will learn how to make frequency tables by tallying in groups of five and write the frequencies. Let us construct frequency tables for a given data.

Example 1. *A group of 30 pupils were surveyed on which animal they would like the most to have as a pet. The results are given below:*

dog, cat, cat, fish, cat, rabbit, dog, cat, rabbit, dog, cat, dog, dog, dog, cat, rabbit, fish, rabbit, dog, cat, dog, cat, cat, dog, rabbit, cat, fish, dog, rabbit, cat.

Make a frequency distribution table for this data.

Solution. For counting purposes, we use tally marks. After putting 4 tally marks vertically, we cross it as shown below and again we take the tally marks in the same manner, counting in sets of fives.

Frequency Distribution Table

Animal	Tally Marks	Number of Animals (Frequency)
Dog	𝍸 𝍸	10
Cat	𝍸 𝍸 |	11
Fish	|||	3
Rabbit	𝍸 |	6
	Total	**30**

As mentioned earlier, the frequency gives the number of times a particular entry occurs. The above table is known as *frequency distribution table* as it gives the number of times an entry occurs.

Example 2. *Consider the following marks (out of 50) obtained in Mathematics by 60 pupils of High School Grade 10.*

21, 10, 30, 22, 33, 5, 37, 12, 25, 42, 15, 39, 26, 32, 18, 27, 28, 19, 29, 35, 31, 24, 36, 18, 20, 38, 22, 44, 16, 24, 10, 27, 39, 28, 49, 29, 32, 23, 31, 21, 34, 22, 23, 36, 24, 36, 33, 47, 48, 50, 39, 20, 7, 16, 36, 45, 47, 30, 22, 17.

Make a frequency table for the above data using intervals 0–10, 10–20 and so on.

Solution. **Grouped Frequency Distribution Table**

Groups	Tally Marks	Frequency
0–10	||	2
10–20	~~||||~~ ~~||||~~	10
20–30	~~||||~~ ~~||||~~ ~~||||~~ ~~||||~~ |	21
30–40	~~||||~~ ~~||||~~ ~~||||~~ ||||	19
40–50	~~||||~~ ||	7
50–60	|	1
	Total	**60**

Data represented in this way is said to be *grouped* and the distribution obtained is called *grouped frequency distribution table.*

It helps us to draw meaningful inferences like:

(*i*) Most of the pupils have scored between 20 and 40.

(*ii*) Eight pupils have scored more than 40 marks out of 50 and so on.

Each of the groups: 0–10, 10–20, 20–30 etc., is called a *Class Interval* (or briefly a class).

Observe that 10 occurs in both classes, *i.e.*, 0–10 as well as 10–20. Similarly, 20 occurs in classes 10–20 and 20–30. But it is not possible that an observation (say 10 or 20) can belong simultaneously to two classes. To avoid this, we adopt the convention that the common observation will belong to the higher class, *i.e.,* 10 belongs to the class interval 10–20 (and not to 0–10).

Similarly, 20 belongs to 20–30 (and not to 10–20). In the class interval, 10–20, 10 is called the *lower class limit* and 20 is called the *upper class limit.*

Similarly, in the class interval 20–30, 20 is the *lower class limit* and 30 is the *upper class limit.*

Observe that the difference between the upper class limit and lower class limit for each of the class intervals 0–10, 10–20, 20–30 etc., is equal, (10 in this case). This difference between the upper class limit and lower class limit is called the *width* or *size* of the class interval.

(*b*) Histograms

A histogram is a graphical representation of a frequency distribution in the form of adjacent rectangles with class intervals as bases and heights proportional to frequency.

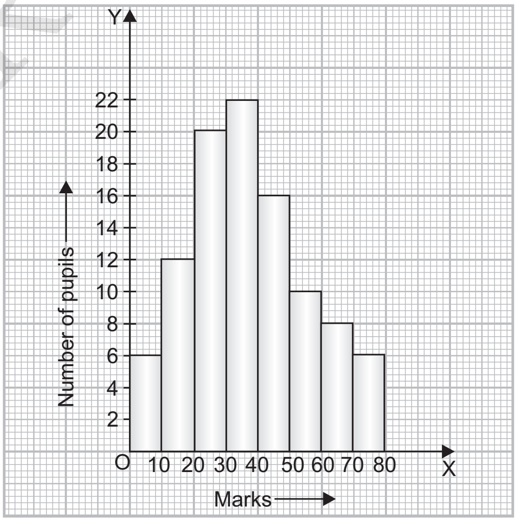
In a histogram, the width of a rectangle is significant and represents the class size.

Example 3. *The following table gives the marks scored by 100 pupils in an entrance examination.*

Marks	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
Number of pupils	6	12	20	22	16	10	8	6

Draw a histogram for the above data.

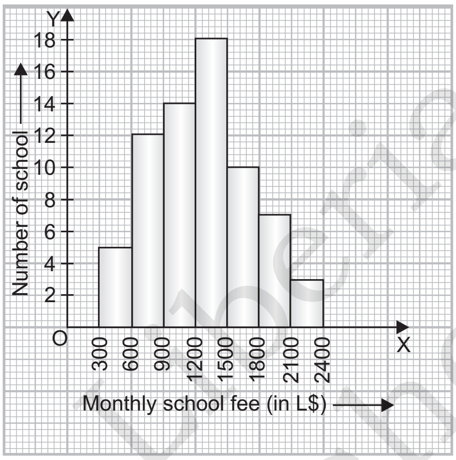
Solution. Represent the class limits along *x*-axis on a suitable scale and the frequencies (*i.e.*, number of pupils) along *y*-axis on a suitable scale. With class intervals as bases and the corresponding frequencies as heights, construct rectangles to obtain the desired histogram.



Example 4. *Construct a histogram for the following data:*

Monthly school fee (in L$)	300-600	600-900	900-1200	1200-1500	1500-1800	1800-2100	2100-2400
Number of school	5	12	14	18	10	7	3

Solution. Represent the class limits along x-axis on a suitable scale and the frequencies (i.e., number of school) along y-axis on a suitable scale. With class intervals as bases and the corresponding frequencies as heights, construct rectangles to obtain the desired histogram (see the figure).



EXERCISE 11.1

1. Carry out simple surveys to collect data for the following:

(*a*) (*i*) Number of pupils below the age of five in the families around you.

(*ii*) Performance of Liberia in football or in athletics

(*iii*) Female literacy rate in a given area

(*iv*) Highest maximum temperature of a city during a year.

(*b*) What kind of data would you need in the above situations? Unless and until you collect appropriate data, you cannot know the desired information.

(*c*) What is the appropriate data for each?

(*d*) Discuss with your friends and identify the data you would need for each. Some of this data is easy to collect and some difficult.

2. Construct frequency tables of the data collected by your classmates, *i.e.*, examination results, rainfall in a month, import and export etc.

3. Construct a frequency distribution table for the data on weights (in kg) of 20 pupils of a class using intervals 30–35, 35–40, and so on.

40, 38, 33, 48, 60, 53, 31, 46, 34, 36, 49, 41, 55, 49, 65, 42, 44, 47, 38, 39.

4. Represent the following data in the form of a histogram:

Class interval	0–10	10–20	20–30	30–40	40–50	50–60	60–70
Frequency	4	10	16	20	15	10	5

5. Draw a histogram to represent the following data:

Class interval	10–15	15–20	20–25	25–30	30–35	35–40
Frequency	30	80	75	55	35	50

11.3 MEASURES OF CENTRAL TENDENCY (MODE, MEDIAN *AND* MEAN)

Different forms of data need different forms of representative or central value to describe it. One of these representative value is '*Mode*'. You will learn about the other representative values '*Median*' and '*Mean*' later on in this section.

Mode

The mode of a set of observations is the observation that occurs most often.

Example 5. *Find the mode of the given set of numbers:*

1, 1, 2, 4, 3, 2, 1, 2, 2, 4.

Solution. On arranging the numbers with same values together, we get

1, 1, 1, 2, 2, 2, 2, 3, 4, 4

Mode of this data is 2 because it occurs the most frequently than other observations.

Example 6. *Following are the margins of victory in the football matches of a league:*

1, 3, 2, 5, 1, 4, 6, 2, 5, 2, 2, 2, 4, 1, 2, 3, 1, 1, 2, 3, 2, 6, 4, 3, 2, 1, 1, 4, 2, 1, 5, 3, 3, 2, 3, 2, 4, 2, 1, 2.

Find the mode of this data.

Solution. Let us put the data in a tabular form:

Margins of Victory	***Tally Bars***	***Number of Matches (Frequency)***
1	𝍸 ||||	9
2	𝍸 𝍸 ||||	14
3	𝍸 ||	7
4	𝍸	5
5	|||	3
6	||	2
	Total	**40**

Looking at the table, we can quickly say that 2 is the '*mode*' since 2 has occurred the highest number of times.

Thus, the most of the matches have been won with a victory margin of 2 goals.

Note: Mode is always one of the given observations.

Median

ACTIVITY 2

Consider a group of 7 pupils with the following heights (in cm):

148, 135, 150, 140, 140, 142, 145

- The games teacher wants to divide the class into two groups.
- Each group has equal number of pupils.
- One group has pupils with height lesser than a particular height and the other group has pupils with heights greater than the particular height.

How would she do that?

Thus, in a given data, arranged in ascending or descending order, the *median* gives us the middle observation. (if the number of observations is odd).

Example 7. *Find the median of the numbers:*

12, 16, 8, 14, 25, 11, 10, 6, 15.

Solution. We first arrange the data in ascending order as follows:

6, 8, 10, 11, **12**, 14, 15, 16, 25

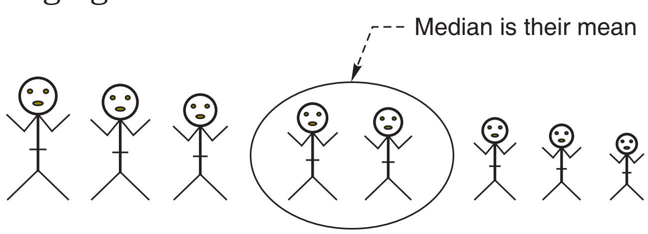
We see that the middle number of these is 12, *i.e.*, four numbers are less than 12 and four numbers are greater than 12.

Hence, the median is 12.

Note: In general, we may not get same value for median and mode.

If there is an even number of observations, the median is found by finding the mean (average) of the two middle values after they have been arranged in ascending or descending order.

The following figure illustrates this.



Observe the following set of numbers:

57, 50, 53, 62, 40, 51, 49, 61

Here, the number of observation is 8, which is even.

The above data can be arranged in descending order as below:

62, 61, 57, **53**, **51**, 50, 49, 40

Here, the two middle values are **53** and **51**.

Thus, median = $\frac{53 + 51}{2} = \frac{104}{2}$ = 52.

Mean

The most common representative value of a group of data is the *arithmetic mean* or the *mean*. To understand this in a better way, let us look at the following Activity:

ACTIVITY 3

Consider two vessels contain 20 litres and 60 litres of milk respectively. What is the amount that each vessel would have, if both share the milk equally?

When we ask this question, we are seeking the arithmetic mean.

In the above case, the average or the arithmetic mean would be

$$\frac{\text{Total quantity of milk}}{\text{Number of vessels}} = \frac{20 + 60}{2} \text{ litres} = 40 \text{ litres.}$$

Thus, each vessel would have 40 litres of milk.

Thus, the average or Arithmetic Mean (A.M.) or simply mean is defined as follows:

$$\text{Mean} = \frac{\text{Sum of all observations}}{\text{Number of observations}}$$

Example 8. *David studies for 4 hours, 5 hours and 3 hours respectively on three consecutive days. How many hours does he study daily on an average?*

Solution. The average study time of David would be

$$\frac{\text{Total number of study hours}}{\text{Number of days for which he studied}}$$

$$= \frac{4 + 5 + 3}{3} \text{ hours} = \frac{12}{3} \text{ hours} = 4 \text{ hours per day.}$$

Thus, we can say that David studies for 4 hours daily on an average.

Note: Mode, Median and Mean of a data may be three different values but always lie between the lowest and the highest observations.

EXERCISE 11.2

1. Find the mode of the following data:
31, 35, 36, 33, 32, 35, 36, 31, 34, 33, 36, 34, 32, 32, 36, 34.

2. Heights (in cm) of 25 children are:
168, 165, 163, 160, 163, 161, 162, 164, 163, 162, 164, 163, 160, 163, 160, 165, 163, 162, 163, 164, 163, 160, 165, 163, 162.
What is the mode of their heights?

3. Your friend found the median and the mode of a given data. Describe and correct your friends error if any:
35, 32, 35, 42, 38, 32, 34
Median = 42, Mode = 32

4. Find the median of the following data:
5, 9, 6, 4, 8, 2, 7

5. Find the median of the following data:
42, 73, 35, 92, 67, 85, 71, 81, 51, 56

6. Find the mean of your study hours or sleeping hours for the whole week.

7. Find the mean attendance of pupils in your school for the previous year.

11.4 STEM *AND* LEAF PLOT

A plot where each data value is split into a "leaf" (usually the last digit) and a "stem" (the other digits) is called a stem and leaf plot.

For example: Observe the following stem and leaf plot:

12, 13, 24, 25, 25, 28, 28, 30, **37**, 49

Stem	**Leaf**				
1	2	3			
2	4	5	5	8	8
3	0	**7**			
4	9				

Here, we have shown how "37" would be split into "3" (stem) and "7" (leaf).

The "*stem*" values are listed down, and the "*leaf*" values are listed next to them.

This way the "*stem*" groups the scores and each "*leaf*" indicates a score within that group.

Example 9. *The heights of 28 pupils, measured to the nearest centimetres, have been found to be as follows:*

161	*150*	*154*	*165*	*168*	*161*	*154*
162	*150*	*151*	*162*	*164*	*171*	*165*
158	*154*	*156*	*172*	*160*	*170*	*153*
159	*161*	*170*	*162*	*165*	*166*	*168*

Represent it using a stem and leaf plot.

Solution. The stem and leaf plot is:

Stem	**Leaf**
15	0 4 4 0 1 8 4 6 3 9
16	1 5 8 1 2 2 4 5 0 1 2 5 6 8
17	1 2 0 0

11.5 GRAPHICAL DISPLAYS

Graphical display is a way of analysing numerical data. It exhibits the relation between data, ideas, information and concepts in a diagram. It is one of the most important learning strategies. It always depends on the type of information in a particular domain. There are different types of graphical displays. Some of them are as follows:

- Line Graphs
- Histograms
- Frequency Table
- Stem and Leaf Plot
- Bar Graphs
- Line Plot
- Circle Graph
- Box and Whisker Plot

Bar Graphs and Double Bar Graphs

Bar graph is used to display the category of data and it compares the data using solid bars to reparesent the quantities.

Example 10. *Maruti Suzuki company has following data of sales recorded for last 3 days.*

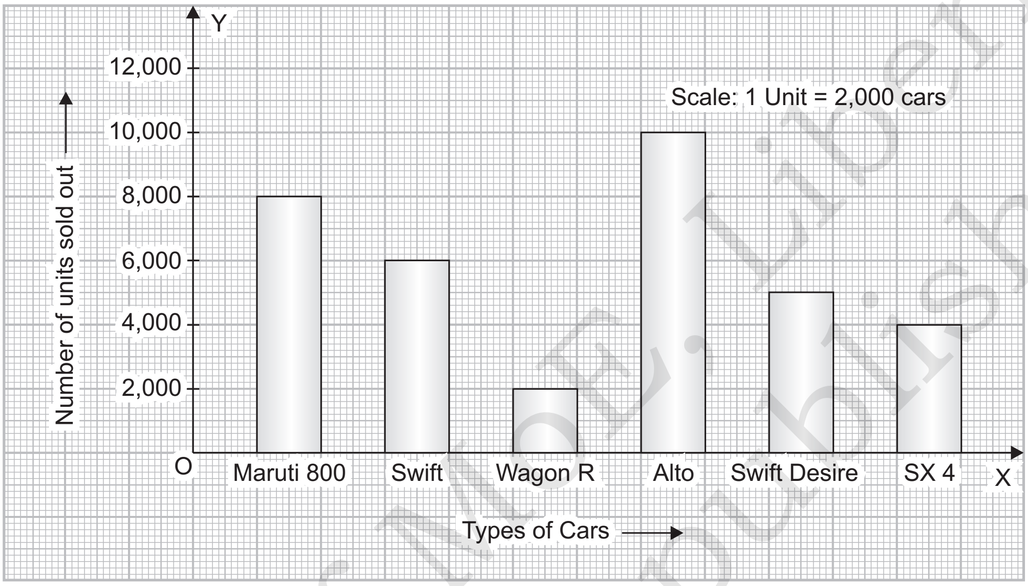
Maruti 800	*Swift*	*Wagon R*	*Alto*	*Swift Desire*	*SX 4*
8,000	*6,000*	*2,000*	*10,000*	*5,000*	*4,000*

Draw a bar graph and answer the following questions:

(*a*) *Which model recorded maximum sales?*

(*b*) *Which model recorded minimum sales?*

Solution. *Bar graph:*



(*a*) Alto model recorded maximum sales.

(*b*) Wagon R model recorded minimum sale.

Example 11. *Following table shows the number of HIV positive data of Liberia (male and female) in the particular years from January 2014 to December 2018.*

Years	*2014*	*2015*	*2016*	*2017*	*2018*
Male	*204*	*224*	*372*	*270*	*339*
Female	*474*	*412*	*685*	*516*	*570*

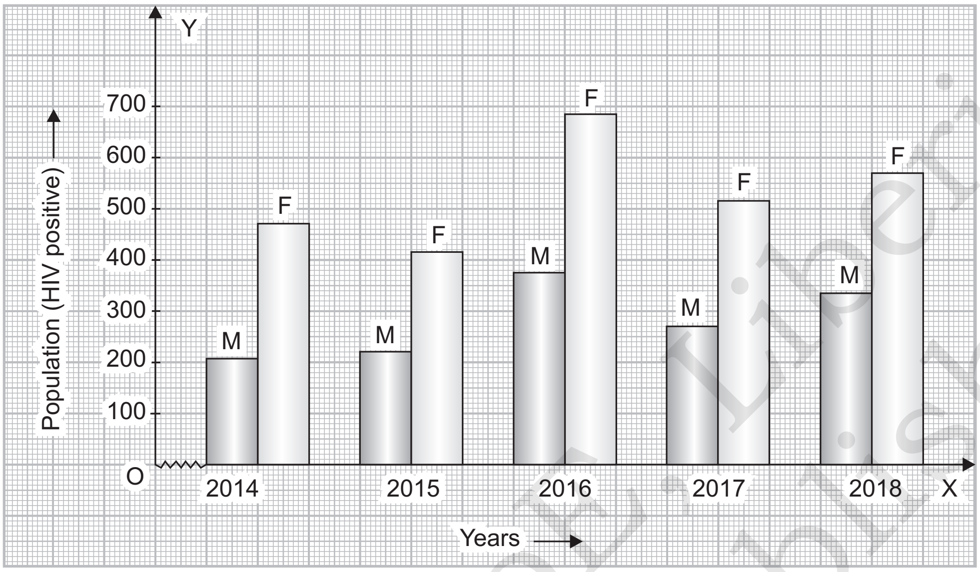
Draw a double bar graph using the above data.

(*a*) *In which year you record the maximum number of patients?*

(*b*) *In which year you record the minimum number of patients?*

Solution. We draw double bars, one bar to show number of HIV +ve cases of male and the other bar to show HIV +ve cases of female year wise.

HIV Positive cases in Liberia



(*a*) In 2016, we record the maximum number of patients.

(*b*) In 2015, we record the minimum number of patients.

Pie Chart

A pie chart shows how something is divided. A pie chart is a circular chart divided into a number of sectors. The circle represents the whole and each sector represents the various observations.

The total angle at the centre of a circle is 360°. The central angle of the sectors will be a fraction of 360°.

Example 12. *The following table shows the sales of some vehicles in a month.*

Name of Vehicle	*Bicycles*	*Scooter*	*Motorcycle*	*Car*	*Bus*	*Total*
Sale in Number	*80*	*40*	*50*	*60*	*10*	*240*

Draw a pie chart to represent the data.

Solution. Since the angle about the centre of the circle is 360°, calculate the corresponding central angles. Here the total sale = 240.

Name of Vehicle	*Sales in number*	*In Fraction*	*Central angle*
Bicycle	80	$\frac{80}{240}$	$\frac{80}{240} \times 360° = 120°$
Scooter	40	$\frac{40}{240}$	$\frac{40}{240} \times 360° = 60°$
Motorcycle	50	$\frac{50}{240}$	$\frac{50}{240} \times 360° = 75°$
Car	60	$\frac{60}{240}$	$\frac{60}{240} \times 360° = 90°$
Bus	10	$\frac{10}{240}$	$\frac{10}{240} \times 360° = 15°$
Total	**240**		**360°**

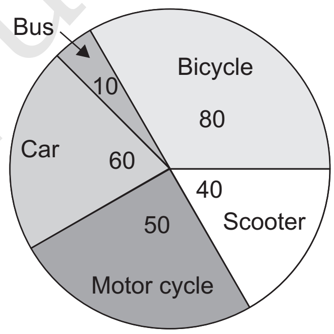
Steps of construction:

(*i*) Draw a circle of any radius.

(*ii*) Draw a horizontal radius of the circle.

(*iii*) Start with horizontal radius to form sectors with central angles of 120°, 60°, 75°, 90° and 15° respectively.

(*iv*) Mention the corresponding vehicles in the sectors.



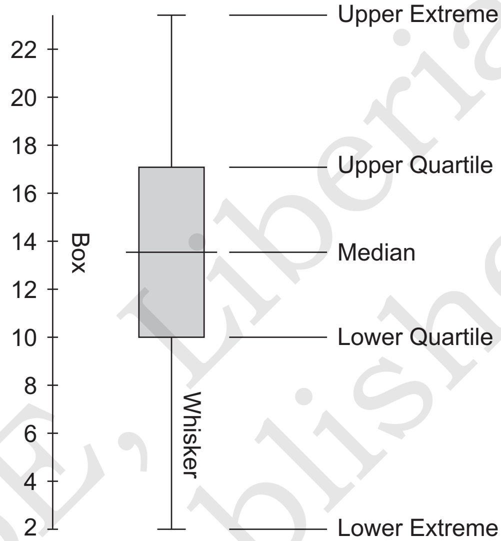
We obtain the required pie chart as shown in the figure.

11.6 BOX *AND* WHISKER PLOT

A box and whisker plot (also known as a *box plot*) is a graph that represents visually data from a five-number summary. These numbers are median, upper and lower quartile, minimum and maximum data value (extremes).

Look at the following example of box and whisker plot:

So, there are a couple of things, you should know in order to work with box plots:



- **Lower Extreme.** The smallest value in a given data set.
- **Upper Extreme.** The highest value in a given data set.
- **Median value.** The middle number in the set.
- **Lower Quartile.** Below that value, the lower 25% of the data are contained.
- **Upper Quartile.** Above that value, the upper 25% of the data are contained.
- **"Whiskers".** The lines that extend from the boxes. They are used to indicate variability out of the upper and lower quartiles.

Example 13. *Draw a simple box and whisker plot to display the data of math test results for a class of 15 pupils. Here are the results:*

91 95 54 69 80 85 88 73 71 70 66 90 86 84 73.

Solution.

Step 1: Order the data points from least to the greatest.

54 66 69 70 71 73 73 80 84 85 86 88 90 91 95

Step 2: Find the median of the data:

54 66 69 70 71 73 73 (80) 84 85 86 88 90 91 95

the median

This is an odd set of data–you have 15 data points. It means the middle point is 80 as there are 7 data points above it and 7 data points below.

Step 3: Find the middle points of the two halves divided by the median (find the upper and lower quartiles).

54 66 69 (70) 71 73 73 (80) 84 85 86 (88) 90 91 95

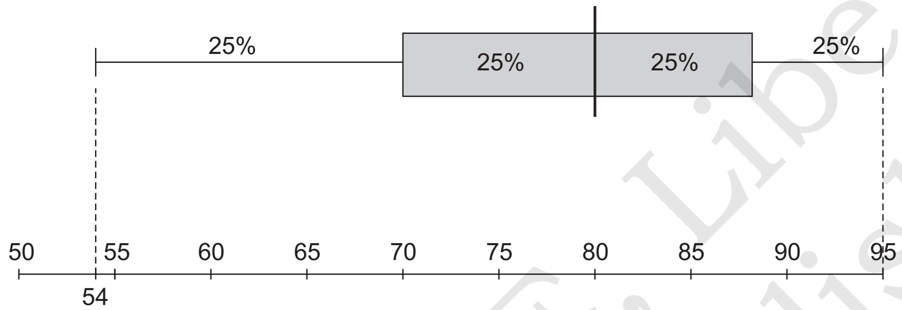
lower quartile upper quartile

Step 4: Find the extreme values.

Extreme value = 54 and 95.

Here the five-number summary for the class of pupils is 54, 70, 80, 88 and 95.

Now draw box-and-whisker plot.



The plot is divided into four groups: a lower whisker, a lower box half, an upper box half, and an upper whisker. Each of those groups shows 25% of the data because we have an equal amount of data in each group.

Interpreting the box and whisker plot results:

The box and whisker plot shows that 50% of the pupils have scores between 70 and 88 points.

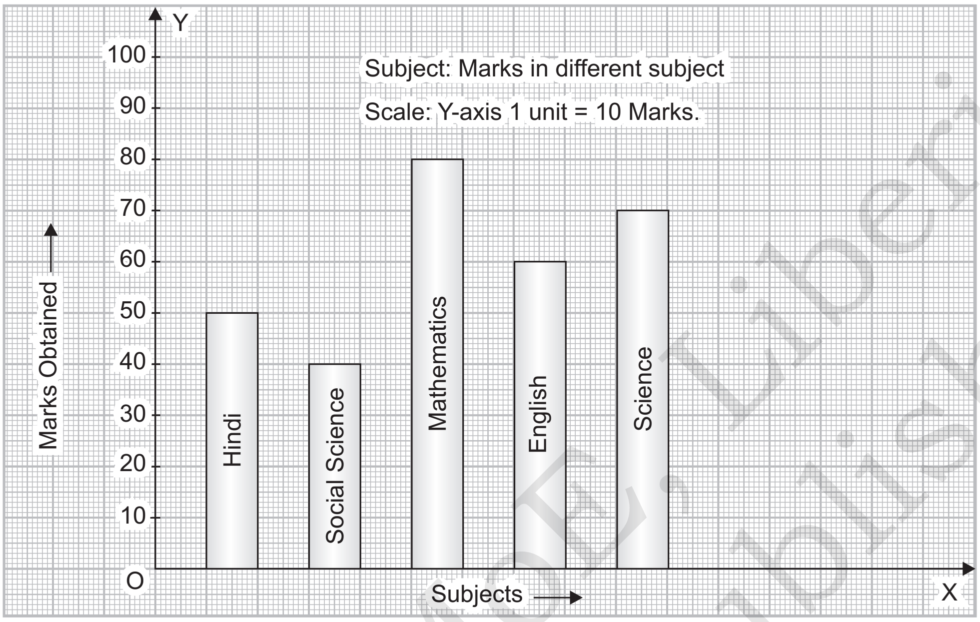
In addition, 75% scored lower than 88 points, and 50% have test results above 80 points.

EXERCISE 11.3

1. Speed of typists, in word per minute are:
76, 75, 80, 78, 75, 80, 80, 75, 76, 76, 79, 76, 77, 76, 79, 80, 79, 78, 77, 79
Draw a stem and leaf plot for this data.

2. Attendances of pupils during a month in a school are:
265, 298, 275, 280, 298, 275, 295, 295, 275, 295, 275, 298, 280, 298, 298, 275, 275, 298, 280, 265, 265, 280
Draw a stem and leaf plot for this data.

3. Read the bar graph given below and answer the following questions.



(*a*) What information does the bar graph give?

(*b*) In which subject the pupil scored maximum marks?

(*c*) In which subject the pupil scored the least marks?

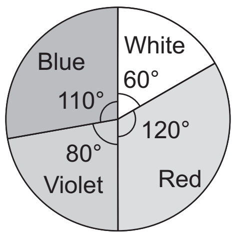
(*d*) In which subject the pupil scored more than English but less than Mathematics?

4. Following table shows population of male and female in some of the cities in a particular year.

Cities population in thousands	Coldwell	Buutuo	Bopolu	Arthington
Male (in thousands)	860	415	315	290
Female (in thousands)	810	390	290	285

Draw a double bar graph.

5. Samuel has 36 coloured marbles shown in the given pie chart. Study it carefully and answer the following questions.

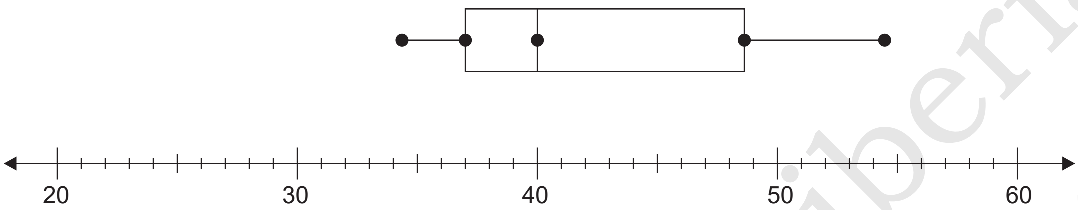


(*a*) Which colour marbles are the most?

(*b*) What is the ratio of white to red marbles Samuel has?

(*c*) How many violet marbles are there?

6. An electronic gadgets distributor distributes various brands of mobiles to retailers. The data for the number of smartphones distributed in nine months (Jan-Sep) are collected to make a box-and-whisker plot. Read the plot and answer the questions.



(*a*) Write the median form the above given plot.
(*b*) What is the least number of smartphones distributed?
(*c*) Write the third quartile from the given plot.

7. Draw the box and whisker plot to display the data: 3, 7, 8, 5, 12, 14, 21, 13, 18.

11.7 AVERAGE

The average of a list of data is the expression of the central value of a set of data. Mathematically, it is defined as the mean value which is equal to the sum of the number of a given set of values divided by the total number of values present in the set. In statistics, the average of a given set of numerical data is known as mean.

For example: The average of 3, 4 and 5 is $\frac{3+4+5}{3} = \frac{12}{3} = 4$

Here, 4 is the central value of 3, 4 and 5.

Thus, the meaning of average is to find the mean value of a group of numbers.

$$\text{Average} = \frac{\text{Sum of Values}}{\text{Number of Values}}$$

Consider, we have n number of values such as $x_1, x_2, x_3, \ldots, x_n$.

Then, average of the given data = $\frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$.

Example 14. *If there are a group of numbers that is 20, 21, 23, 22, 21, 20, 23, then find the average of these numbers.*

Solution. Since, average = $\frac{\text{Sum of Values}}{\text{Number of Values}}$

$$\therefore \quad \text{Average} = \frac{20+21+23+22+21+20+23}{7} = \frac{150}{7} = 21.42$$

Example 15. *Find the average of 3, –7, 6, 12, –2.*

Solution. Sum of the numbers = 3 + (–7) + 6 + 12 + (– 2)

= 3 – 7 + 6 + 12 – 2

= 21 – 9 = 12

Number of numbers = 5

$\therefore$ Average = $\frac{12}{5}$ = 2.4.

EXERCISE 11.4

1. What is average? Give an example.
2. What is the average formula?
3. Is the average and mean of numbers same?
4. What is the average of 10, 15, 20, 100?
5. Find the average of first five prime numbers.
6. If the heights of males in a group are: 5.5, 5.3, 5.7, 5.9, 6, 5.10, 5.8, 5.6, 5.4, 6. Then find the average height.

B. RATIO *AND* RATES

11.8 RATIO

A ratio is a comparison of two or more similar quantities. It expresses what part of one is contained in the other. It has no unit. If a and b are two similar quantities. Then a/b or $a : b$ (read as a is to b) is called their ratio. The quantities a and b are known as terms of the ratio. The first term is known as *antecedent* and the second term is known as consequent.

Note: A ratio is not changed by multiplying or dividing the antecedent and the consequent by the same non zero number.

Example 16. *Find the value of x for which*

(*a*) $x + 7 : x + 4 = 3 : 2$ (*b*) $5x + 15 : 2x + 3 = 10 : 3$

Solution. (*a*) Given $\frac{x+7}{x+4} = \frac{3}{2}$

$\Rightarrow \quad 2(x + 7) = 3(x + 4) \quad \Rightarrow \quad 2x + 14 = 3x + 12$

$\Rightarrow \quad 14 - 12 = 3x - 2x \quad \Rightarrow \quad x = 2$

(*b*) Given $\frac{5x+15}{2x+3} = \frac{10}{3}$

$\Rightarrow \quad 3(5x + 15) = 10(2x + 3) \quad \Rightarrow \quad 15x + 45 = 20x + 30$

$\Rightarrow \quad 45 - 30 = 20x - 15x \quad \Rightarrow \quad 15 = 5x$

$\Rightarrow \quad x = 3$

Example 17. *Express each ratio as a fraction in its lowest form.*

(*a*) *25 cm : 3 m* (*b*) *48 s : 5 minutes*

Solution. (*a*) Using 1 m = 100 cm, we have 3 m = 300 cm. Therefore, the given ratio becomes 25 cm : 300 cm or $\frac{25}{300}$ or $\frac{1}{12}$.

(*b*) Using 1 minute = 60 s, we have 5 minutes = 300 s. Therefore, the given ratio becomes 48 s : 300 s or $\frac{48}{300}$ or $\frac{12}{75}$ or $\frac{4}{25}$.

11.9 SCALE DRAWING

You know that a map is a reduced represntation of a very large region. A scale is usually given at the top or bottom of every map. The scale shows a relationship between actual length and the length represented on the map. The scale of a map is thus the ratio of the distance between two points on the map to the actual distance between those two points on land (the ground). The ratio is often in the form 1 : *n*.

Note: A *scale* is used to reduced the actual distance between two points on a map.

If the scale of a map is 1 : *n*, then;

1. The *distance on the ground or real length*

$= n \times$ distance on the map.

2. The *distance on the map*

$$= \frac{\text{distance on the ground or real length}}{n}$$

Example 18. *The height of a tower of a church building in a scale drawing is 2 cm. If the scale is 1 cm to 20 m. How tall is the actual tower?*

Solution. Let actual height of the tower = h cm.

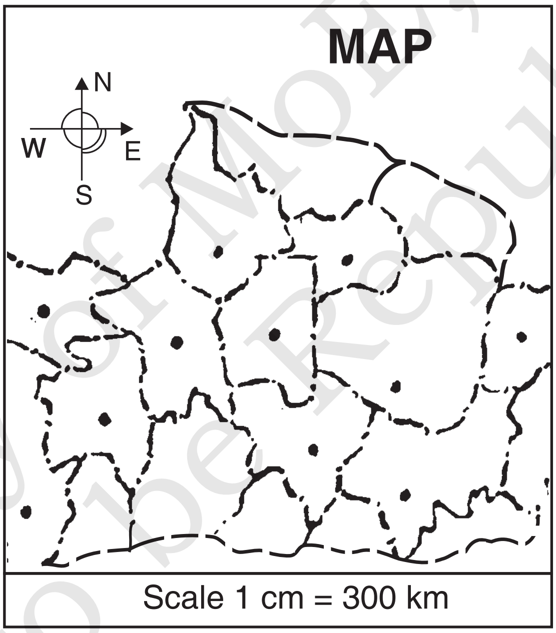
$\therefore$ 20 m = 2000 cm [$\because$ 1 m = 100 cm]

then $1 : 2000 = 2 : h$ *i.e.,* $\frac{1}{2000} = \frac{2}{h}$

$\Rightarrow$ $h = 2 \times 2000$ $\Rightarrow$ $h = 4000$

So, actual height of the tower = 4000 cm = $\frac{4000}{100}$ = 40 m.

Example 19. *The scale of a map is given as 1 : 30000000. Two cities are 4 cm apart on the map. Find the actual distance between them.*



Solution. Let the map distance be x cm and actual distance be y cm, then

$$1 : 30000000 = x : y \Rightarrow \frac{1}{3 \times 10^7} = \frac{x}{y}$$

Since $x = 4$ so, $\frac{1}{3 \times 10^7} = \frac{4}{y}$

$\Rightarrow$ $y = 4 \times 3 \times 10^7 = 12 \times 10^7$ cm = 1200 km.

Thus, two cities, which are 4 cm apart on the map, are actually 1200 km away from each other.

EXERCISE 11.5

1. Find the ratio between 10 kg and 15 kg.
2. Find which is greater
   (*a*) 2 : 3 or 3 : 4 (*b*) 5 : 7 or 7 : 9 (*c*) 1 : 4 or 2 : 9
3. Find the value of x for which $5x + 1 : 2x + 3 = 1 : 2$.
4. Two numbers are in the ratio 1 : 2. When 4 is added to each, the ratio becomes 2 : 3. Find the numbers.
5. Monthly incomes of two persons are in the ratio of 4 : 5. Their monthly expenditures are in the ratio of 7 : 9. If each saves L$ 50 a month, find their monthly incomes.
6. A map is drawn to a scale of 1 : 20000.
   (*a*) Find the distance on the map between two points that are 12 km apart.
   (*b*) The distance on the map between two villages is 25 cm. Find the real distance between the two villages.
7. A map of a large town is drawn to the scale of 1 : 100000. What is the distance in kilometres (km) represented by a line segment 4 cm on the ground?
8. The length of a field 1.2 km long is represented on a map by a line 40 mm long. What is the scale of the map?

11.10 RATES

A rate is a ratio that is used for comparing two different kinds of quantities which have different units.

For example: If there is a comparison of books to pupils. These are comparing two different units (an object and a person). If the rate is 4 : 1, this indicates that there are 4 books for 1 pupil.

Also, if William drives 300 miles in 3 hours. Then this is a rate. It can be converted to a *unit rate* by dividing by the second value (the denominator) when each number is divided by 3, a unit rate of 100 miles for every 1 hour or 100 miles per hour is calculated,

i.e., $$\frac{300}{3} : \frac{3}{3} = \frac{100}{1}$$

In fact, unit rate is said to be the amount of something in each unit or per unit.

Note: **1.** A rate is a special kind of ratio that is used to show the comparison of two different units of measurement while a ratio compares the same units.

2. A rate is a ratio because it compares two numbers, yet a ratio cannot be a rate because it only compares the same units.

Example 20. *A man travels 20 kilometers in 2 hours, find the unit rate.*

Solution. In 2 hours a man travels 20 kilometers

In 1 hour the man will travel = $\frac{20}{2}$ = 10 kilometer

∴ The required unit rate = 10 km/hour.

Example 21. *If a car uses 2 hours to consume 10 litres of petrol, find its unit rate and then find in*

(*a*) *5 hours* (*b*) *30 minutes*

Solution. In 2 hours the car consume petrol = 10 litres

In 1 hour the car will consume petrol = $\frac{10}{2}$ = 5 litres

∴ Unit rate = 5 litres per hour

(*a*) In 5 hours the car will consume petrol = 5 × 5 = 25 litres

(*b*) In 30 minutes = $\frac{30}{60}$ hour = $\frac{1}{2}$ hour

The car will consume petrol = $\frac{1}{2} \times 5$ = 2.5 litres.

11.11 TRAVEL GRAPHS USING RATES

Travel Graphs are the line graphs which are used to describe the motion of objects (such as bicycle, car, train, bike etc).

1. A *distance-time graph* shows the relationship between the distance (from a fixed point) and the time of a journey. The distance travelled is represented on the vertical axis and the time taken to travel the distance is represented on the horizontal axis.

2. A *speed-time graph* shows the relationship between the speed and the time of a Journey. The speed is represented on the vertical axis and the time taken to travel on the horizontal axis.

Reading a travel graph. In distance-time graph, the speed of an object at any instant is given by

$$\text{Speed} = \frac{\text{Distance travalled}}{\text{Time taken}};$$

$$\text{Average speed} = \frac{\text{Total distance travelled}}{\text{Total time taken}}$$

Gradient: The gradient of a straight line joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

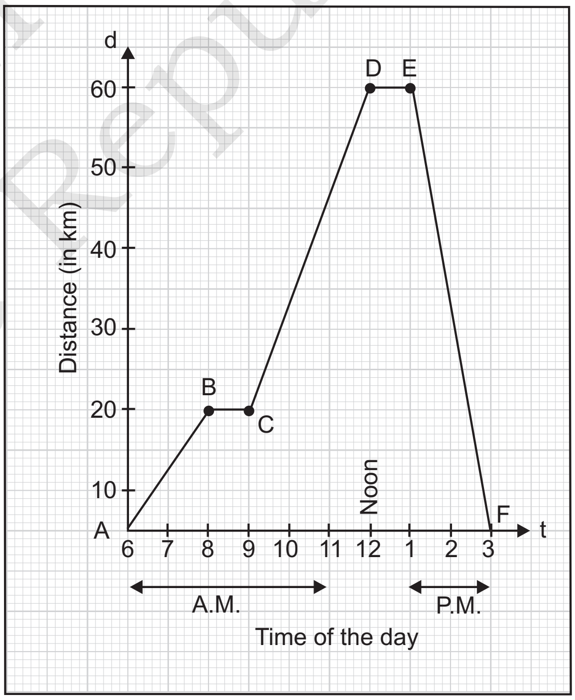
$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$

The *gradient* of a distance-time graph represents the *speed*.

The gradient of a speed-time graph represents the *acceleration*.

Note: The gradient of a horizontal line is always zero.

Example 22. *Consider the distance time graph of a cyclist as shown in the figure.*



Answer the following questions:

(*a*) *When the cyclist leave home?*

(*b*) *When did the cyclist return home?*

(*c*) *How far away from home was he at 9 a.m.?*

(*d*) *How far away from home was he at noon?*

(*e*) *How far away from was he at 1 p.m.?*

(*f*) *At what times did he take a rest?*

(*g*) *How far away from home was he at 2 p.m.?*

(*h*) *How far away from home was he at 3 p.m.?*

(*i*) *How far away from home was he at 8 a.m.?*

(*j*) *Find his speed from:*

(*i*) *6 a.m. to 8 a.m.*

(*ii*) *12 noon to 1 p.m.*

(*iii*) *1 p.m. to 3 p.m.*

(*iv*) *When was the cyclist travelling most quickly?*

Solution. Looking at the graph,

(*a*) The cyclist left home at 6 a.m.

(*b*) The cyclist returned home at 3 p.m.

(*c*) At 9 a.m., he was 20 km away from his home.

(*d*) At noon (i.e., at 12), he was 60 km away from his home.

(*e*) At 1 p.m., he was again 60 km away from his home.

(*f*) He took rest between noon and 1 p.m.

(*g*) At 2 p.m., he was 35 km away from his home.

(*h*) At 3 p.m., he reached his home, so he was zero km away from his home.

(*i*) At 8 a.m. he was 20 km away from his home.

(*j*) (*i*) Let A(6 a.m., 0 km) and B(8 a.m., 20 km) represent the coordinates of A and B respectively.

∴ Required speed = gradient of the straight line AB

$$= \frac{20\text{ km} - 0\text{ km}}{8\text{ a.m.} - 6\text{ a.m.}} = \frac{20}{2}\text{ km/hr}$$

$$= 10\text{ km/hr}$$

(*ii*) Let D(12 (noon), 60 km) and E (1 p.m.; 60 km) represent the coordinates of D and E respectively.

∴ Required speed = gradient of the straight line DE

$$= \frac{60\text{ km} - 60\text{ km}}{1\text{ p.m.} - 12\text{ (noon)}} = 0\text{ km/hr}$$

(*iii*) Let the coordinates of E and F are E(1 p.m., 60 km) and F(3 p.m., 0 km) respectively.

∴ Required speed = gradient of the straight line EF

$$= \frac{0\text{ km} - 60\text{ km}}{3\text{ p.m.} - 1\text{ p.m.}} = -\frac{60}{2}\text{ km/hr}$$

$$= -30\text{ km/hr}$$

Here negative sign indicates the motion in opposite direction. Thus, the cyclist is moving towards home at a speed of 30 km/hr.

(*iv*) Using part *j*(*i*), *j*(*ii*) and *j*(*iii*), we observe that the cyclist was travelling the most quickly between 1 p.m. and 3 p.m.

11.12 CONVERSION GRAPHS USING RATES

Conversion graphs are straight line graphs that show a relationship between two units. It can be used to convert from one to another.

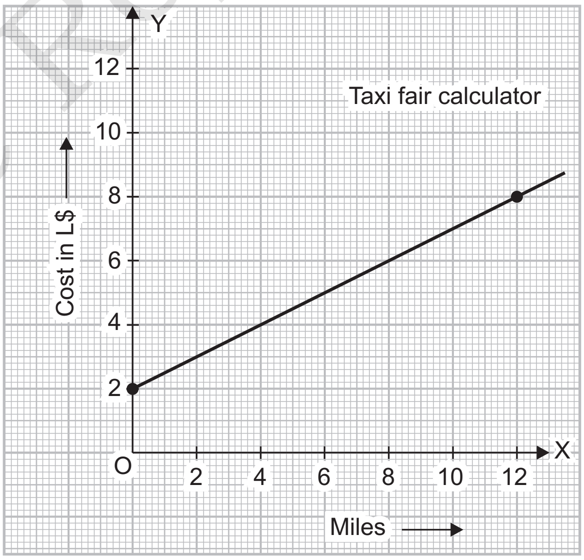
In order to draw a conversion graph:

1. Draw axes and choose which units should be represented by the horizontal and vertical.

2. Use information provided to plot points that represent the conversion between the units.

3. Draw a straight line through the points ensuring it is long enough to span the width and height of the axes.

Example 23. *Daniel's taxi calculate the cost of a journey using the following conversion graph.*



Read the graph and answer the following questions:

(*i*) *What is the fix rate of the journey?*

(*ii*) *If a journey cost L$ 10, how many miles would you expect to have travelled? You should show your answer on the graph.*

(*iii*) *Calculate the cost of a 30 mile journey.*

(*iv*) *Calculate the distance travelled when the journey costs L$ 15.*

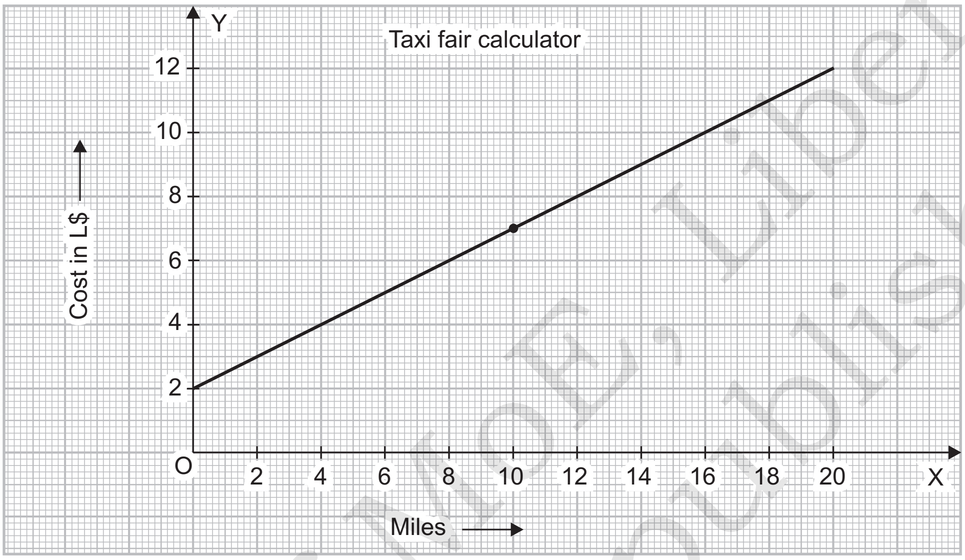
Solution. (*i*) From the graph, 0 mile = L$ 2.

This is a flat rate regardless of the distance travelled

∴ The fix rate is L$ 2.

(*ii*) As the graph doesn't pass through (0, 0) to undertake a conversion that is outside the scale on the graph a different approach must be taken.

It may be possible to extend the graph further.



We can observe from this extanded graph that L$10 would be the charge for a journey that is 16 miles long.

Note: If it is not possible to extend the graph, we will need to understand some calculations.

(*iii*) We know that a 0 miles journey costs L$ 2

We know that a 10 miles journy costs L$ 7

So each 10 miles travelled costs L$ 7 – L$ 2 = L$ 5

30 miles = 10 miles × 3, so that cost of 30 miles = L$ 5 × 3 = L$ 15

Add on the flat charge of L$ 2 for each journey:

L$ 15 + L$ 2 = L$ 17.

(*iv*) From the graph we can see that L$ 2 = 0 mile and L$ 3 = 2 miles.

This means that after the flat rate of L$ 2, L$ 1 is added on for every 2 miles.

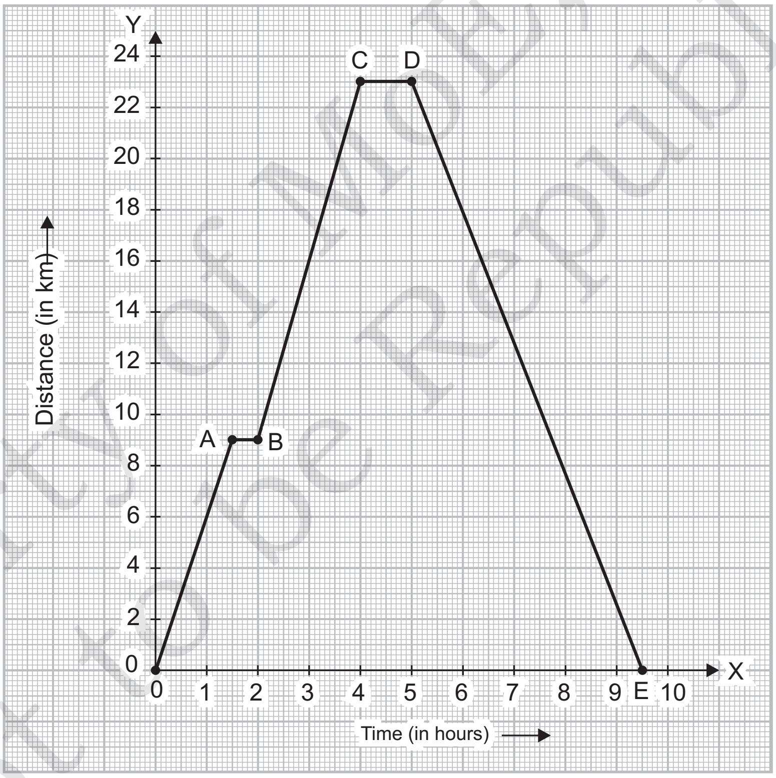
For a journey costing L$ 15, we can subtract the flat rate of L$ 2 to see that L$ 13 has been added on for distance travelled.

As each L$ 1 accounts for 2 miles: 13 × 2 = 26 miles.

EXERCISE 11.6

1. What is the difference between rate and ratio?
2. Ella earns L$ 30000 working 15 hours. Find the unit rate.
3. A man travels 400 kilometers in 10 hours, find the unit rate.
4. Henry can type 200 words in 8 minutes. Working at the same rate, how long does he take to type 800 words?
5. 15 boys can do a piece of work in 60 days. Working at the same rate, how many boys will be required to complete the same work in 20 days?
6. The graph in the figure shows the journey of Albert during a sponsored walk. Describe the parts of the journey shown by:

   (*a*) OA (*b*) AB (*c*) BC (*d*) CD (*e*) DE



7. The graph in the figure shows the journey made by two people, Lisa and Daniel.

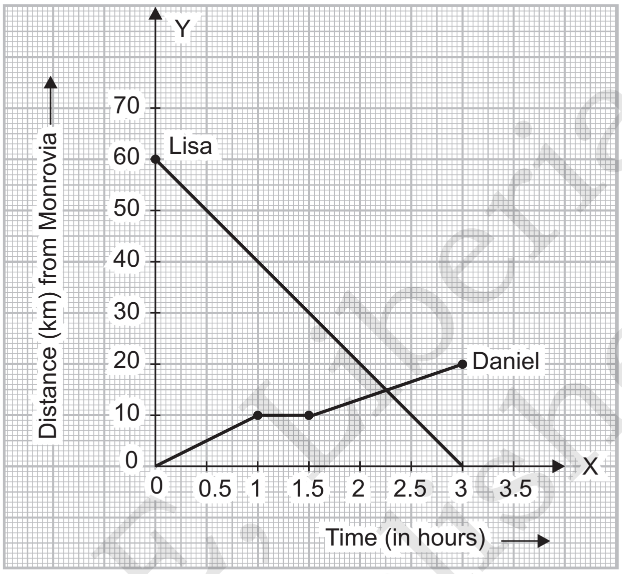
   Daniel sets out from Monrovia at mid-day to travel to Caldwell. At the same time, Lisa sets off from Caldwell and travels to Monrovia at a constant speed. From the graph answer the following questions.

(*i*) What is the distance between Monrovia and Caldwell?

(*ii*) Which traveller is travelling by car and which by cycle? How did you decide?

(*iii*) At what time and where do they meet?

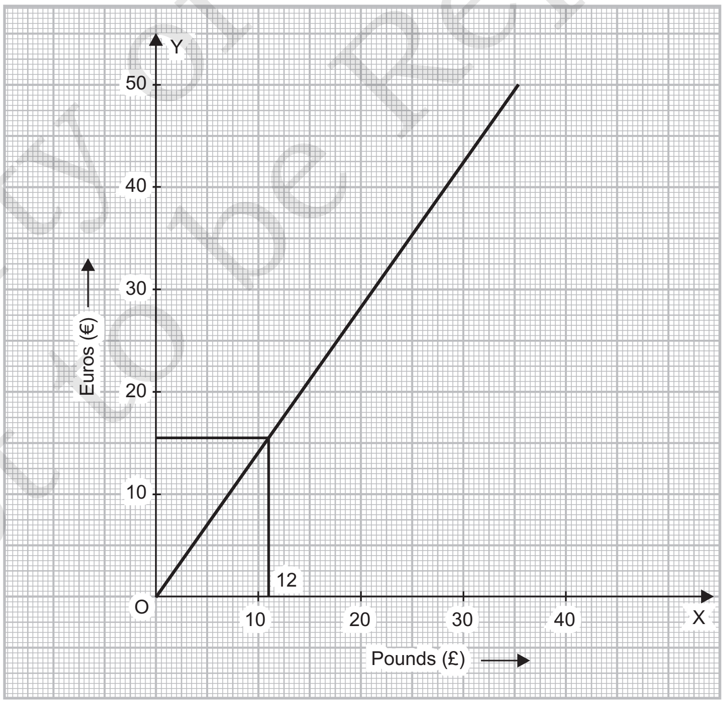
(*iv*) What is Lisa's average speed on the journey?



8. The conversion graph can be used to change between pound (£) and Euros (€).

(*a*) Use the graph to change 30 Pounds to Euros.

(*b*) Use the graph to change 16 Euros to Pounds.

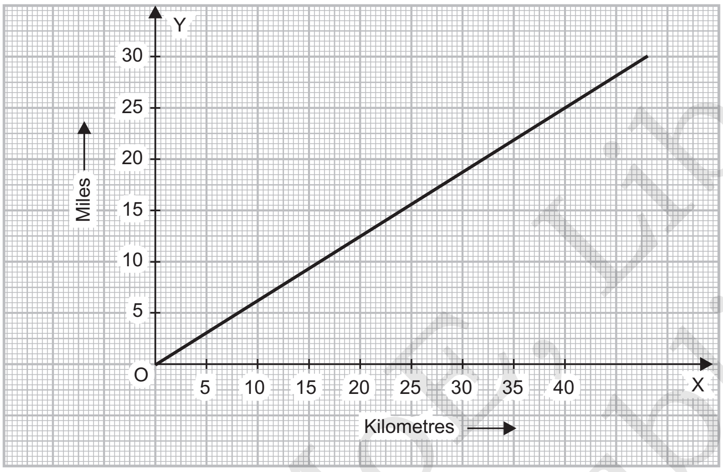


9. A conversion graph for kilometers and miles is shown.

(*a*) Use the graph to convert 40 kilometres to miles.

(*b*) Use the graph to convert 10 miles to kilometres.

(*c*) Convert 200 kilometres to miles.



10. Use the following information to sketch a conversion graph

°C	0	60	100
°F	0	140	212

(*a*) Use graph to change 100°F to °C.

(*b*) Use graph to change 76°C to °F.

(*c*) Convert 200°C to °F.

C. PERCENTAGES

11.13 PERCENTAGE

The word *'percentage'* means 'per hundred' or 'out of hundred'. It is denoted by the symbol % (read as 'percent').

For example: 37 percent = 37% = 37 out of 100 = $\frac{37}{100}$

- **Converting a percentage into a fraction**

  Drop the percentage sign (%) and divide by 100

  For example: $53\% = \frac{53}{100}$

- **Converting a fraction into a percentage**

  Multiply the fraction by 100 and put the percentage sign %.

  For example:

$$\frac{a}{b} = \left(\frac{a}{b}\times 100\right)\%$$

$$\frac{4}{5} = \left(\frac{4}{5}\times 100\right)\% = 80\%$$

- **Converting a ratio into a percentage**

  First convert the ratio into a fraction and then convert the fraction into a percentage.

  For example:

$$a : b = \frac{a}{b} = \left(\frac{a}{b}\times 100\right)\%$$

$$5 : 8 = \frac{5}{8} = \left(\frac{5}{8}\times 100\right)\% = \frac{125}{2}\% = 62.5\%$$

- **Percentage of a quantity**

$$r\% \text{ of } x = \frac{r}{100}\times x$$

  For example:

$$16\% \text{ of 25 litres} = \frac{16}{100}\times 25 \text{ litres} = 4 \text{ litres}$$

- **One quantity as a percentage of another quantity**

  To compare two quantities a and b (in same units), we express one as a percentage of the other. Thus,

$$a = \frac{a}{b} \text{ of } b = \left(\frac{a}{b}\times 100\right)\% \text{ of } b$$

$$b = \frac{b}{a} \text{ of } a = \left(\frac{b}{a}\times 100\right)\% \text{ of } a$$

- **Percentage increase/decrease in a quantity**

$$\text{Percentage increase} = \left(\frac{\text{Increase in quantity}}{\text{Original quantity}}\times 100\right)\%$$

$$\text{Percentage decrease} = \left(\frac{\text{Decrease in quantity}}{\text{Original quantity}}\times 100\right)\%$$

- **Increase by x%**

  New (increased) quantity

$$= \text{Original quantity} + \text{Increase in quantity}$$
$$= \text{Original quantity} + x\% \text{ of original quantity}$$
$$= \text{Original quantity} + \frac{x}{100} \text{ of original quantity}$$
$$= \left(1 + \frac{x}{100}\right) \text{ of original quantity}$$

Note: The above result is useful in problems on growth.

- **Decrease by x%**

  New (decreased) quantity

$$= \text{Original quantity} - \text{Decrease in quantity}$$
$$= \text{Original quantity} - x\% \text{ of original quantity}$$
$$= \text{Original quantity} - \frac{x}{100} \text{ of original quantity}$$
$$= \left(1 - \frac{x}{100}\right) \text{ of original quantity}$$

Note: The above result is useful in problems on decay, depreciation etc.

Percentage error: Where there is an approximation, there is an error. The percentage error in measuring two numbers is defined as % error

$$= \frac{\text{Difference between two numbers}}{\text{Original number}} \times 100$$
$$= \frac{\text{Error}}{\text{Original number}} \times 100$$

where error = difference between two numbers.

Example 24. *Express 18 hours as a percentage of 3 days.*

Solution. Since 1 day = 24 hours, we have 3 days = 3 × 24 = 72 hours

We have to express 18 hours as a percentage of 72 hours.

Required percentage = $\left(\frac{18}{72} \times 100\right)\% = 25\%$

Example 25. *The size of a bag that could hold 5 kg of sugar has now been increased so that it can hold 6 kg. What is the percentage increase in size?*

Solution. Original capacity of bag = 5 kg

New capacity of bag = 6 kg

$\Rightarrow$ Increase in capacity = 6 kg – 5 kg = 1 kg

Percentage increase in capacity

$$= \left(\frac{\text{Increase in capacity}}{\text{Original capacity}} \times 100\right)\%$$

$$= \left(\frac{1}{5} \times 100\right)\% = 20\%.$$

Example 26. *A number is increased by 10% and then it is decreased by 10%. Find the net increase or decrease percentage.*

Solution. Let the original number be 100. It is increased by 10%. Therefore,

$$\text{Increased number} = \left(1 + \frac{10}{100}\right) \times 100 = \frac{11}{10} \times 100 = 110$$

This number is decreased by 10%

$$\therefore \quad \text{Decreased number} = \left(1 - \frac{10}{100}\right) \times 110 = \frac{9}{10} \times 110 = 99$$

which is less than the original number.

Net decrease = 100 – 99 = 1

Therefore, net percentage decrease

$$= \left(\frac{\text{Net decrease}}{\text{Original number}} \times 100\right)\%$$

$$= \left(\frac{1}{100} \times 100\right)\% = 1\%$$

EXERCISE 11.7

1. A man spends 92% of his monthly income. If he saves L$ 220 per month, what is his monthly income?
2. The value of a machine depreciates every year by 10%. What will be its value after 2 years if its present value is L$ 50,000?
3. Samuel used the value of p = 3.14 in a certain calculation whereas the actual value of p is 3.141592654. Find the percentage error up to 2 decimal places.

REVIEW EXERCISE

1. A Junior Football Club in Monrovia has thirty members. Their ages are given below. Make a frequency distribution table for it.

13, 17, 13, 13, 14, 15, 15, 14, 16, 14, 16, 17, 16, 16, 13, 15, 16, 15, 15, 14, 14, 15, 15, 13, 13, 15, 14, 13, 16, 17.

2. Represent the following data in form of a histogram.

Class interval	10–20	20–30	30–40	40–50	50–60	60–70
Frequency	1	5	4	8	5	2

3. The scores in mathematics test (out of 25) of 15 pupils is as follows:

19, 25, 23, 20, 9, 20, 15, 10, 5, 16, 25, 20, 24, 12, 20.

Find the mode and median of this data. Are they same?

4. The heights of 5 pupils in a group are:

152 cm, 170 cm, 156 cm, 164 cm and 158 cm

(*a*) Find the mean height

(*b*) How many pupils have heights more than the mean height?

5. Find the mode of the following data:

20, 21, 25, 22, 17, 22, 13, 15, 23, 21, 9, 10, 22, 20, 30.

6. Find the median of the following data:

2, 5, 3, 2, 4, 5, 2, 4, 6, 8, 7, 9

7. Find the mean temperature of a city/village for the last month.

8. Find the mean of 6.5, 8.2, 9.4, 4.6, 7.8 and 4.9.

9. The heights (in cm) of 20 pupils is given below:

106, 110, 123, 125, 117, 120, 112, 115, 110, 120, 115, 102, 115, 115, 109, 107, 115, 101, 108, 129.

Represent the above data using the stem and leaf plot.

10. The number of pupils passed in Grade-10 (Semester-II) in two consecutive years is as under.

Standard	V	VI	VII	VIII	IX	X
No. of pupils passed in 2020	90	100	115	115	110	100
No. of pupils passed in 2021	80	95	100	105	110	95

Draw a double bar graph using the above data.

11. The number of pupils who opted for coaching classes in various activities organised by the school were as follow:

Activity	Cricket	Photography	Karate	Music	Dance
No. of pupils	75	20	30	45	10

Represent this data on a pie chart.

12. If the age of 9 pupils in a team is 12, 13, 11, 12, 13, 12, 11, 12, 12. Then find the average age of pupils in the team.

13. Find the average of first four multiples of 2.

14. Find the average of 6, 13, 17, 21, 23.

15. Two numbers are in the ratio of 3 : 4. When 8 is subtracted from each, the ratio becomes 2 : 3. Find the numbers.

16. A uniform rod of length 8 m has mass 20 kg. What is the mass per metre?

17. A tap leaks at a rate of 2 cm^3 per second. How long will it take to fill a container of 45 litres capacity (1 litre = 1000 cm^3).

18. Juliet measured the length of her classroom and obtained 4.99 m with a percentage error of 5%. Her own measurement was smaller than the original length. What was the actual length of the room?

MULTIPLE CHOICE QUESTIONS (MCQs)

1. The mean of the numbers 4, 3, 3, x is 5. Find x.

(a) 20 (b) 10 (c) 5 (d) 4

2. The ages in years of eight boys are 14, 14.5, 12, 11.5, 15, 13, 10.5, 13.5. What is their average age?

(a) 14 (b) 13 (c) 12 (d) 11

Directions: *The marks obtained by 10 boys in a test are 0, 1, 3, 3, 5, 7, 8, 9, 9, 9. Use this information to answer questions 3 to 4.*

3. Find the median score.

(a) 3 (b) 5 (c) 6 (d) 7

4. Calculate the mean score.

(a) 4.4 (b) 5.4 (c) 6 (d) 6.4

5. The marks obtained by six boys in a test are 20, 25, 15, 30, 28 and 16. Find the mean mark.

(a) 19.33 (b) 22.33 (c) 23.20 (d) 26.40

6. The ratio 9 : x is equivalent to 36 : 20. What is the value of x?

(a) 4 (b) 5 (c) 6 (d) 8

7. The angles of a triangle are in the ratio 3 : 2 : 1. Find the value of the smallest angle.

(a) 30° (b) 45° (c) 60° (d) 90°

8. A map of a large town is drawn to the sacle of 1 : 100,000. What is the distance in kilometres (km) represented by a line segment 4 cm long on the map?

(a) 0.04 km (b) 0.4 km (c) 4 km (d) .40 km

9. Ten pupils in Monrovia High School took 9 days to weed the school compound. How long would 15 pupils take to weed the compound if they worked at the same rate?

(a) 5 days (b) 6 days (c) $13\frac{1}{2}$ days (d) 14 days

10. The scale of a map is 1 : 100,000. What is the distance (in kilometres) between two towns 4 cm apart on the map?

(a) 0.04 (b) 0.4 (c) 4.0 (d) 400

11. A tank contains 250 litres of water. If 96 litres is used, what percentage of the original quantity is left?

(a) 61.6% (b) 60.5% (c) 59.0% (d) 54.2%

12. Ella obtained 150 marks out of 240 marks in an English test. What was her percentage score?

(a) 33.33% (b) 36.5% (c) 41.67% (d) 62.5%

RECAP AT A GLANCE

- Statistics is the branch of Mathematics which deals with the collection, presentation and analysis of numerical data
- The mode of a set of observations is the observation that occurs most often.
- Median refers to the value which lies in the middle of the data.
- A plot where each data value is split into a "leaf" and a "stem" is called a stem and leaf plot.
- The average of a list of data is the expression of the central value of a set of data.
- A ratio is a comparison of two or more similar quantities.
- A rate is a ratio that is used for comparing two different kinds of quantities which have different units.
- Conversion graphs are straight line graphs that show a relationship between two units.
- The word *'percentage'* means 'per hundred' or 'out of hundred'. It is denoted by the symbol % (read as 'percent').

❑ ❑ ❑

TABLE

Tangents of Angles															*Tan θ°*
θ°	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	*ADD Differences*				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1′	2′	3′	4′	5′
0	.0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	5	9	12	14
1	.0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	5	9	12	14
2	.0349	0367	0384	0402	0419	0437	0454	0472	0489	0507	3	5	9	12	14
3	.0524	0542	0559	0577	0594	0612	0629	0647	0664	0682	4	5	9	12	14
4	.0699	0717	0734	0752	0769	0787	0805	0822	0840	0857	4	5	9	12	14
5	.0875	0892	0910	0928	0945	0963	0981	0998	1016	1033	4	5	9	12	14
6	.1051	1069	1086	1104	1122	1139	1157	1175	1192	1210	4	5	9	12	14
7	.1228	1246	1263	1281	1299	1317	1334	1352	1370	1388	4	5	9	12	14
8	.1405	1423	1441	1459	1477	1495	1512	1530	1548	1566	4	5	9	12	14
9	.1584	1602	1620	1638	1655	1673	1691	1709	1727	1745	4	5	9	13	14
10	.1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	4	5	9	13	14
11	.1944	1962	1980	1998	2016	2035	2053	2071	2089	2107	4	5	9	13	15
12	.2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	4	5	9	13	15
13	.2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	4	6	9	13	15
14	.2493	2512	2530	2549	2568	2586	2605	2623	2642	2661	4	6	9	14	15
15	.2679	2698	2717	2736	2754	2773	2792	2811	2830	2849	4	6	9	14	15
16	.2867	2886	2905	2924	2943	2962	2981	3000	3019	3038	4	6	9	14	15
17	.3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	4	6	10	14	15
18	.3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	4	6	10	14	15
19	.3443	3463	3482	3502	3522	3541	3561	3581	3600	3620	4	6	10	14	15
20	.3640	3659	3679	3699	3719	3739	3759	3779	3799	3819	4	6	10	14	16
21	.3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	4	6	10	14	16
22	.4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	4	6	10	15	17
23	.4245	4265	4286	4307	4327	4348	4369	4390	4411	4431	4	6	10	15	17
24	.4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	4	6	11	15	17
25	.4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	4	6	11	15	17
26	.4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	4	7	11	16	17
27	.5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	4	7	11	16	18
28	.5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	5	7	11	16	18
29	.5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	5	7	12	16	18
30	.5774	5797	5820	5844	5867	5890	5914	5938	5961	5985	5	7	12	17	19
31	.6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	5	7	12	17	19
32	.6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	5	7	12	18	20
33	.6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	5	8	13	18	20
34	.6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	5	8	13	18	21
35	.7002	7028	7054	7080	7107	7133	7159	7186	7212	7239	5	8	13	19	21
36	.7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	6	8	14	19	22
37	.7536	7563	7590	7618	7646	7673	7701	7729	7757	7785	6	8	14	20	22
38	.7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	6	9	14	21	23
39	.8089	8127	8156	8185	8214	8243	8273	8302	8332	8361	6	9	15	21	23
40	.8391	8421	8451	8481	8511	8541	8571	8601	8632	8662	6	9	15	22	24
41	.8693	8724	8754	8785	8816	8847	8878	8910	8941	8972	6	9	16	22	25
42	.9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	6	9	16	23	26
43	.9325	9358	9391	9424	9457	9490	9523	9556	9590	9623	7	10	17	24	27
44	.9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	7	10	17	25	27
45	1.0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	7	10	18	25	28

Tangents of Angles — *Tan θ°*

θ°	0′ 0.0°	6′ 0.1°	12′ 0.2°	18′ 0.3°	24′ 0.4°	30′ 0.5°	36′ 0.6°	42′ 0.7°	48′ 0.8°	54′ 0.9°	ADD Differences 1′	2′	3′	4′	5′
46	1.0355	0392	0428	0464	0501	0538	0575	0612	0649	0686	7	11	18	26	29
47	1.0724	0761	0799	0837	0875	0913	0951	0990	1028	1067	8	11	19	27	31
48	1.1106	1145	1184	1224	1263	1303	1343	1383	1423	1463	8	12	20	28	32
49	1.1504	1544	1585	1626	1667	1708	1750	1792	1833	1875	8	12	21	29	33
50	1.1918	1960	2002	2045	2088	2131	2174	2218	2261	2305	9	13	22	30	35
51	1.2349	2393	2437	2482	2527	2572	2617	2662	2708	2753	9	14	23	32	36
52	1.2799	2846	2892	2938	2985	3032	3079	3127	3175	3222	9	14	23	32	36
53	1.3270	3319	3367	3416	3465	3514	3564	3613	3663	3713	10	15	25	35	39
54	1.3764	3814	3865	3916	3968	4019	4071	4124	4176	4229	10	16	26	36	41
55	1.4281	4335	4388	4442	4496	4550	4605	4659	4715	4770	11	16	27	38	44
56	1.4826	4882	4938	4994	5051	5108	5166	5224	5282	5340	11	17	29	40	46
57	1.5399	5458	5517	5577	5637	5697	5757	5818	5880	5941	12	18	30	42	48
58	1.6003	6066	6128	6191	6255	6319	6383	6447	6512	6577	13	19	32	45	51
59	1.6643	6709	6775	6842	6909	6977	7045	7113	7182	7251	14	20	34	47	54
60	1.7321	7391	7461	7532	7603	7675	7747	7820	7893	7966	14	22	36	50	58
61	1.8040	8115	8190	8265	8341	8418	8495	8572	8650	8728	15	23	38	54	61
62	1.8807	8887	8967	9047	9128	9210	9292	9375	9458	9542	16	25	41	57	66
63	1.9626	9711	9797	9883	9970	2.006	2.015	2.023	2.032	2.041	18	26	44	61	70
64	2.0503	0594	0686	0778	0872	0965	1060	1155	1251	1348	19	28	47	66	75
65	2.1445	1543	1642	1742	1842	1943	2045	2148	2251	2355	20	30	51	67	81
66	2.2460	2566	2673	2781	2889	2998	3109	3220	3332	3445	22	33	55	77	88
67	2.3559	3673	3789	3906	4023	4142	4262	4383	4504	4627					
68	2.4751	4876	5002	5129	5257	5386	5517	5649	5782	5916					
69	2.6051	6187	6325	6464	6605	6746	6889	7034	7179	7326					
70	2.7475	7625	7776	7929	8083	8239	8397	8556	8716	8878					
71	2.9042	9208	9375	9544	9714	9887	3.006	3.024	3.042	3.060					
72	3.0777	0961	1146	1334	1524	1716	1910	2106	2305	2506					
73	3.2709	2914	3122	3332	3544	3759	3977	4197	4420	4646					
74	3.4874	5105	5339	5576	5816	6059	6305	6554	6806	7062					
75	3.7321	7583	7848	8118	8391	8667	8947	9232	9520	9812					
76	4.0108	0408	0713	1022	1335	1653	1976	2303	2635	2972					
77	4.3315	3662	4015	4374	4737	5107	5483	5864	6252	6646					
78	4.7046	7453	7867	8288	8716	9152	9594	5.005	5.050	5.097					
79	5.1446	1929	2422	2924	3435	3955	4486	5026	5578	6140					
80	5.6713	7297	7894	8502	9124	9758	6.041	6.107	6.174	6.243					
81	6.3138	3859	4596	5350	6122	6912	7720	8548	9395	7.026					
82	7.1154	2066	3002	3962	4947	5958	5996	8062	9158	8.029					
83	8.1443	2636	3863	5126	6427	7769	9152	9.058	9.205	9.357					
84	9.514	9.677	9.845	10.02	10.20	10.39	10.58	10.78	10.99	11.20					
85	11.43	11.66	11.91	12.16	12.43	12.71	13.00	13.30	13.62	13.95					
86	14.30	14.67	15.06	15.46	15.89	16.35	16.83	17.34	17.89	18.46					
87	19.08	19.74	20.45	21.20	22.02	22.90	23.86	24.90	26.03	27.27					
88	28.64	30.14	31.82	33.69	35.80	38.19	40.92	44.07	47.74	52.08					
89	57.29	63.66	71.62	81.85	95.49	114.6	143.2	191.0	286.5	573.0					